4

INTEGRATION

4.1 THE DEFINITE INTEGRAL

We shall begin our study of the integral calculus in the same way in which we began with the differential calculus—by asking a question about curves in the plane.

Suppose f is a real function continuous on an interval I and consider the curve $y = f(x)$. Let $a < b$ where a, b are two points in I, and let the curve be above the x-axis for x between a and b; that is, $f(x) \geq 0$. We then ask: What is meant by the *area* of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$? That is, what is meant by the area of the shaded region in Figure 4.1.1? We call this region the region under the curve $y = f(x)$ between a and b.
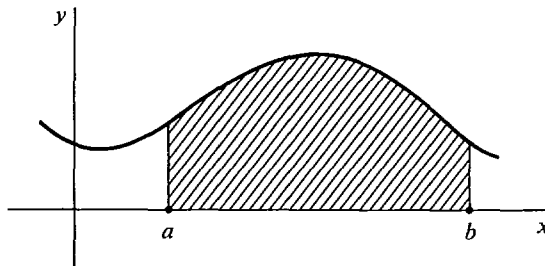

Figure 4.1.1 The Region under a Curve

The simplest possible case is where f is a constant function; that is, the curve is a horizontal line $f(x) = k$, where k is a constant and $k \geq 0$, shown in Figure 4.1.2. In this case the region under the curve is just a rectangle with height k and width $b - a$, so the area is defined as

$$\text{Area} = k \cdot (b - a).$$

The areas of certain other simple regions, such as triangles, trapezoids, and semicircles, are given by formulas from plane geometry.
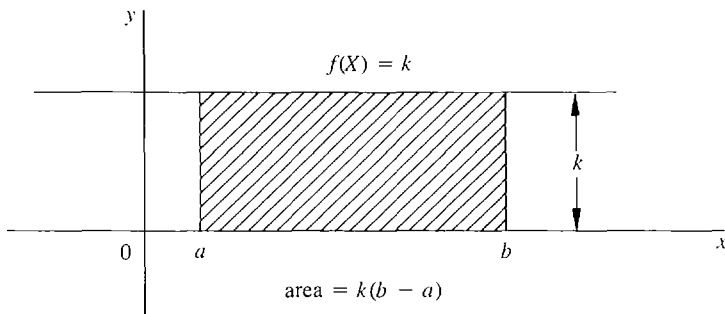
Figure 4.1.2

The area under any continuous curve $y = f(x)$ will be given by the definite integral, which is written

$$\int_a^b f(x)\,dx.$$

Before plunging into the detailed definition of the integral, we outline the main ideas.

First, the region under the curve is divided into infinitely many vertical strips of infinitesimal width dx. Next, each vertical strip is replaced by a vertical rectangle of height $f(x)$, base dx, and area $f(x)\,dx$. The next step is to form the sum of the areas of all these rectangles, called the infinite Riemann sum (look ahead to Figures 4.1.3 and 4.1.11). Finally, the integral $\int_a^b f(x)\,dx$ is defined as the standard part of the infinite Riemann sum.

The infinite Riemann sum, being a sum of rectangles, has an infinitesimal error. This error is removed by taking the standard part to form the integral.

It is often difficult to compute an infinite Riemann sum, since it is a sum of infinitely many infinitesimal rectangles. We shall first study finite Riemann sums, which can easily be computed on a hand calculator.

Suppose we slice the region under the curve between a and b into thin vertical strips of equal width. If there are n slices, each slice will have width $\Delta x = (b - a)/n$. The interval $[a, b]$ will be partitioned into n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

where $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_n = b.$

The points $x_0, x_1, \ldots, x_n$ are called *partition points*. On each subinterval $[x_{k-1}, x_k]$, we form the rectangle of height $f(x_{k-1})$. The kth rectangle will have area

$$f(x_{k-1}) \cdot \Delta x.$$

From Figure 4.1.3, we can see that the sum of the areas of all these rectangles will be fairly close to the area under the curve. This sum is called a *Riemann sum* and is equal to

$$f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x.$$

It is the area of the shaded region in the picture. A convenient way of writing Riemann sums is the "Σ-notation" (Σ is the capital Greek letter sigma),

$$\sum_a^b f(x)\,\Delta x = f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x.$$
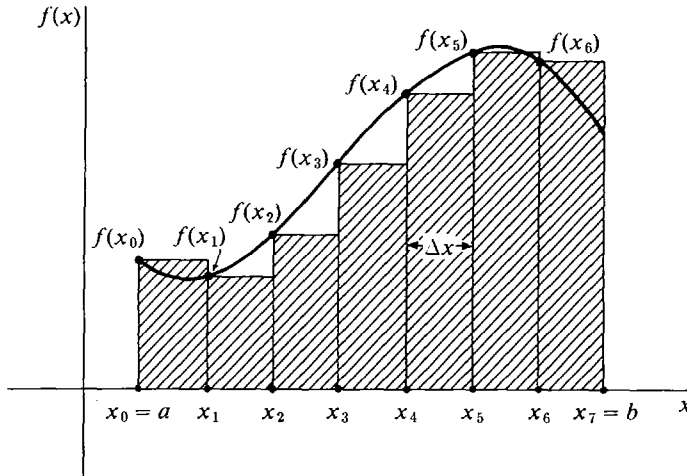
Figure 4.1.3 The Riemann Sum

The a and b indicate that the first subinterval begins at a and the last subinterval ends at b.

We can carry out the same process even when the subinterval length Δx does not divide evenly into the interval length $b - a$. But then, as Figure 4.1.4 shows, there will be a remainder left over at the end of the interval $[a, b]$, and the Riemann sum will have an extra rectangle whose width is this remainder. We let n be the largest integer such that

$$a + n \Delta x \leq b,$$

and we consider the subintervals

$$[x_0, x_1], \ldots, [x_{n-1}, x_n], [x_n, b],$$

where the partition points are

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2 \Delta x, \ldots, \quad x_n = a + n \Delta x, \quad b.$$
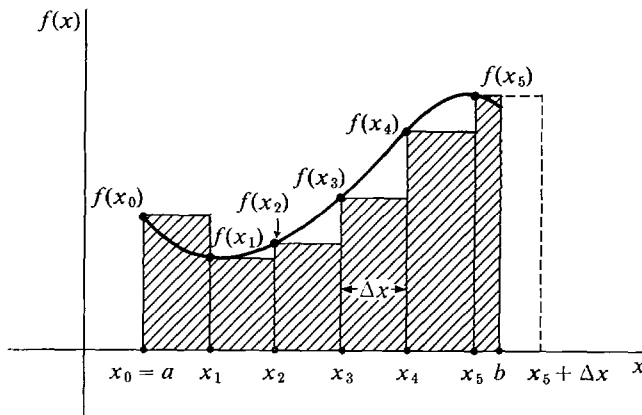

Figure 4.1.4

x_n will be less than or equal to b but $x_n + \Delta x$ will be greater than b. Then we define the *Riemann sum* to be the sum

$$\sum_a^b f(x)\, \Delta x = f(x_0)\, \Delta x + f(x_1)\, \Delta x + \cdots + f(x_{n-1})\, \Delta x + f(x_n)(b - x_n).$$

Thus given the function f, the interval $[a, b]$, and the real number $\Delta x > 0$, we have defined the Riemann sum $\sum_a^b f(x)\, \Delta x$. We repeat the definition more concisely.

DEFINITION

> Let $a < b$ and let Δx be a positive real number. Then the **Riemann sum** $\sum_a^b f(x)\, \Delta x$ is defined as the sum
>
> $$\sum_a^b f(x)\, \Delta x = f(x_0)\, \Delta x + f(x_1)\, \Delta x + \cdots + f(x_{n-1})\, \Delta x + f(x_n)(b - x_n)$$
>
> where n is the largest integer such that $a + n\, \Delta x \le b$, and
>
> $$x_0 = a, \quad x_1 = a + \Delta x, \cdots, \quad x_n = a + n\, \Delta x, \quad b$$
>
> are the partition points.

If $x_n = b$, the last term $f(x_n)(b - x_n)$ is zero. The Riemann sum $\sum_a^b f(x)\, \Delta x$ is a real function of three variables a, b, and Δx,

$$\sum_a^b f(x)\, \Delta x = S(a, b, \Delta x).$$

The symbol x which appears in the expression is called a *dummy variable* (or *bound variable*), because the value of $\sum_a^b f(x)\, \Delta x$ does not depend on x. The dummy variable allows us to use more compact notation, writing $f(x)\, \Delta x$ just once instead of writing $f(x_0)\, \Delta x, f(x_1)\, \Delta x, f(x_2)\, \Delta x$, and so on.

From Figure 4.1.5 it is plausible that by making Δx smaller we can get the Riemann sum as close to the area as we wish.
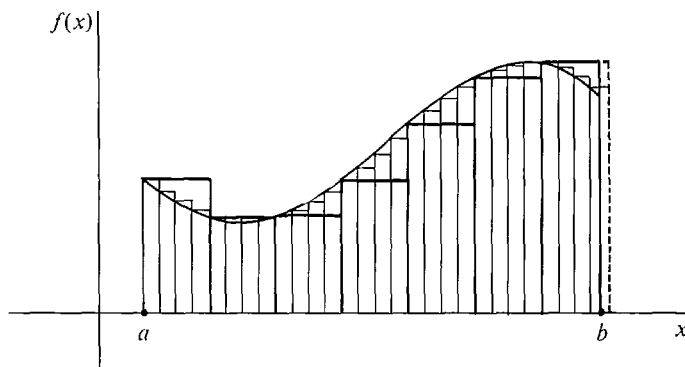

Figure 4.1.5

EXAMPLE 1   Let $f(x) = \tfrac{1}{2}x$. In Figure 4.1.6, the region under the curve from $x = 0$ to $x = 2$ is a triangle with base 2 and height 1, so its area should be
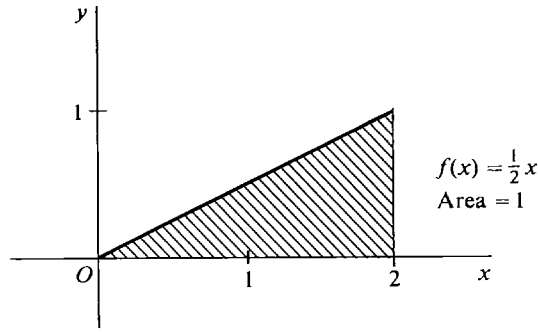
$$A = \tfrac{1}{2}bh = 1.$$

Figure 4.1.6

Let us compare this value for the area with some Riemann sums. In Figure 4.1.7, we take $\Delta x = \frac{1}{2}$. The interval $[0, 2]$ divides into four subintervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$, and $[\frac{3}{2}, 2]$. We make a table of values of $f(x)$ at the *lower* endpoints.

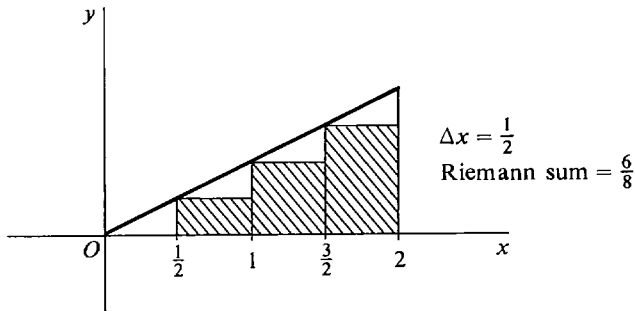| x_k	0	$\frac{1}{2}$	1	$\frac{3}{2}$
$f(x_k)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$

Figure 4.1.7

The Riemann sum is then

$$\sum_0^2 f(x)\,\Delta x = 0 \cdot \tfrac{1}{2} + \tfrac{1}{4} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{3}{4} \cdot \tfrac{1}{2} = \tfrac{6}{8}.$$

In Figure 4.1.8, we take $\Delta x = \frac{1}{4}$. The table of values is as follows.

x_k	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$
$f(x_k)$	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$

The Riemann sum is

$$\sum_0^2 f(x)\,\Delta x = 0 \cdot \tfrac{1}{4} + \tfrac{1}{8} \cdot \tfrac{1}{4} + \tfrac{2}{8} \cdot \tfrac{1}{4} + \tfrac{3}{8} \cdot \tfrac{1}{4} + \tfrac{4}{8} \cdot \tfrac{1}{4} + \tfrac{5}{8} \cdot \tfrac{1}{4} + \tfrac{6}{8} \cdot \tfrac{1}{4} + \tfrac{7}{8} \cdot \tfrac{1}{4} = \tfrac{7}{8}.$$

We see that the value is getting closer to one.

Finally, let us take a value of Δx that does not divide evenly into the interval length 2. Let $\Delta x = 0.6$. We see in Figure 4.1.9 that the interval then divides into three subintervals of length 0.6 and one of length 0.2, namely $[0, 0.6]$, $[0.6, 1.2]$, $[1.2, 1.8]$, $[1.8, 2.0]$.

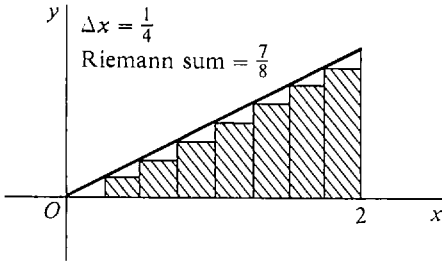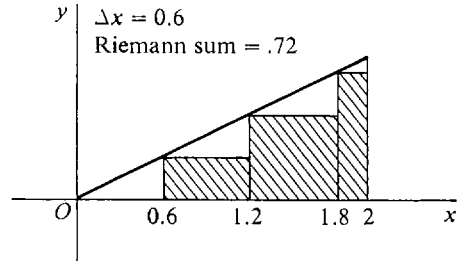| x_k	0	0.6	1.2	1.8
$f(x_k)$	0	0.3	0.6	0.9

Figure 4.1.8

Figure 4.1.9

The Riemann sum is

$$\sum_{0}^{2} f(x)\,\Delta x = 0(.6) + (.3)(.6) + (.6)(.6) + (.9)(.2) = .72.$$

EXAMPLE 2 Let $f(x) = \sqrt{1 - x^2}$, defined on the closed interval $I = [-1, 1]$. The region under the curve is a semicircle of radius 1. We know from plane geometry that the area is $\pi/2$, or approximately $3.14/2 = 1.57$. Let us compute the values of some Riemann sums for this function to see how close they are to 1.57. First take $\Delta x = \frac{1}{2}$ as in Figure 4.1.10(a). We make a table of values.

x_k	-1	$-1/2$	0	$1/2$
$f(x_k)$	0	$\sqrt{3/4}$	1	$\sqrt{3/4}$

The Riemann sum is then

$$\sum_{-1}^{1} f(x)\,\Delta x = 0 \cdot 1/2 + \sqrt{3/4} \cdot 1/2 + 1 \cdot 1/2 + \sqrt{3/4} \cdot 1/2$$

$$= \frac{1 + \sqrt{3}}{2} \sim 1.37.$$

Next we take $\Delta x = \frac{1}{5}$. Then the interval $[-1, 1]$ is divided into ten subintervals as in Figure 4.1.10(b). Our table of values is as follows.

x_k	-1	$-\dfrac{4}{5}$	$-\dfrac{3}{5}$	$-\dfrac{2}{5}$	$-\dfrac{1}{5}$	0	$\dfrac{1}{5}$	$\dfrac{2}{5}$	$\dfrac{3}{5}$	$\dfrac{4}{5}$
$f(x_k)$	0	$\dfrac{3}{5}$	$\dfrac{4}{5}$	$\dfrac{\sqrt{21}}{5}$	$\dfrac{\sqrt{24}}{5}$	1	$\dfrac{\sqrt{24}}{5}$	$\dfrac{\sqrt{21}}{5}$	$\dfrac{4}{5}$	$\dfrac{3}{5}$

(a)

(b)
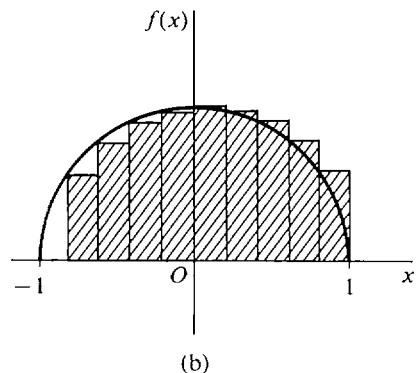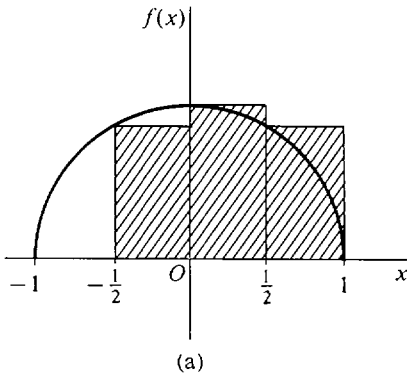
Figure 4.1.10

The Riemann sum is

$$\sum_{-1}^{1} f(x)\,\Delta x = \frac{1}{5}\left[0 + \frac{3}{5} + \frac{4}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{24}}{5} + 1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{4}{5} + \frac{3}{5}\right]$$

$$= \frac{19 + 2\sqrt{21} + 2\sqrt{24}}{25} \sim 1.52.$$

Thus we are getting closer to the actual area $\pi/2 \sim 1.57$.

By taking Δx small we can get the Riemann sum to be as close to the area as we wish.

Our next step is to take Δx to be *infinitely small* and have an *infinite* Riemann sum. How can we do this? We observe that if the real numbers a and b are held fixed, then the Riemann sum

$$\sum_{a}^{b} f(x)\,\Delta x = S(\Delta x)$$

is a real function of the single variable Δx. (The symbol x which appears in the expression is a dummy variable, and the value of

$$\sum_{a}^{b} f(x)\,\Delta x$$

depends only on Δx and not on x.) Furthermore, the term

$$\sum_{a}^{b} f(x)\,\Delta x = S(\Delta x)$$

is defined for all real $\Delta x > 0$. Therefore by the Transfer Principle,

$$\sum_{a}^{b} f(x)\,dx = S(dx)$$

is defined for all hyperreal $dx > 0$. When $dx > 0$ is infinitesimal, there are infinitely many subintervals of length dx, and we call

$$\sum_{a}^{b} f(x)\,dx$$

an *infinite Riemann sum* (Figure 4.1.11).

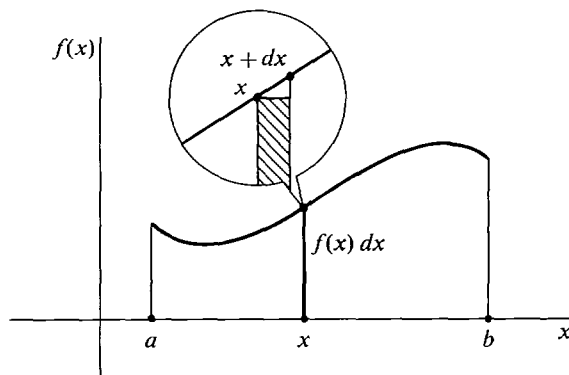**Figure 4.1.11**   Infinite Riemann Sum

We may think intuitively of the Riemann sum

$$\sum_a^b f(x)\, dx$$

as the infinite sum

$$f(x_0)\, dx + f(x_1)\, dx + \cdots + f(x_{H-1})\, dx + f(x_H)(b - x_H)$$

where H is the greatest hyperinteger such that $a + H\, dx \le b$. (Hyperintegers are discussed in Section 3.8.) H is positive infinite, and there are $H + 2$ partition points $x_0, x_1, \ldots, x_H, b$. A typical term in this sum is the infinitely small quantity $f(x_K)\, dx$ where K is a hyperinteger, $0 \le K < H$, and $x_K = a + K\, dx$.

The infinite Riemann sum is a hyperreal number. We would next like to take the standard part of it. But first we must show that it is a finite hyperreal number and thus has a standard part.

THEOREM 1

Let f be a continuous function on an interval I, let $a < b$ be two points in I, and let dx be a positive infinitesimal. Then the infinite Riemann sum

$$\sum_a^b f(x)\, dx$$

is a finite hyperreal number.

PROOF Let B be a real number greater than the maximum value of f on $[a, b]$. Consider first a real number $\Delta x > 0$. We can see from Figure 4.1.12 that the
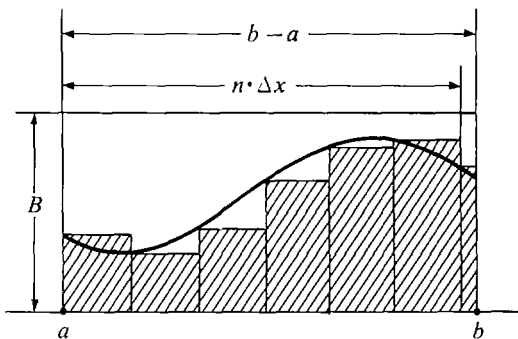

Figure 4.1.12

finite Riemann sum is less than the rectangular area $B \cdot (b - a)$;

$$\sum_a^b f(x)\, \Delta x < B \cdot (b - a).$$

Therefore by the Transfer Principle,

$$\sum_a^b f(x)\, dx < B \cdot (b - a).$$

In a similar way we let C be less than the minimum of f on $[a, b]$ and show that

$$\sum_a^b f(x)\,dx > C \cdot (b-a).$$

Thus the Riemann sum $\quad \sum_a^b f(x)\,dx \quad$ is finite.

We are now ready to define the central concept of this chapter, the definite integral. Recall that the derivative was defined as the standard part of the quotient $\Delta y/\Delta x$ and was written dy/dx. The "definite integral" will be defined as the standard part of the infinite Riemann sum

$$\sum_a^b f(x)\,dx,$$

and is written $\int_a^b f(x)\,dx$. Thus the Δx is changed to dx in analogy with our differential notation. The Σ is changed to the long thin S, i.e., $\int$, to remind us that the integral is obtained from an infinite sum. We now state the definition carefully.

DEFINITION

*Let f be a continuous function on an interval I and let $a < b$ be two points in I. Let dx be a positive infinitesimal. Then the **definite integral** of f from a to b with respect to dx is defined to be the standard part of the infinite Riemann sum with respect to dx, in symbols*

$$\int_a^b f(x)\,dx = st\left(\sum_a^b f(x)\,dx\right).$$

We also define
$$\int_a^a f(x)\,dx = 0,$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

By this definition, for each positive infinitesimal dx the definite integral

$$\int_u^w f(x)\,dx$$

is a real function of two variables defined for all pairs (u, w) of elements of I. The symbol x is a dummy variable since the value of

$$\int_u^w f(x)\,dx$$

does not depend on x.

In the notation $\sum_a^b f(x)\,dx$ for the Riemann sum and $\int_a^b f(x)\,dx$ for the integral, we always use matching symbols for the infinitesimal dx and the dummy variable x. Thus when there are two or more variables we can tell which one is the dummy variable in an integral. For example, $x^2 t$ can be integrated from 0 to 1 with respect to either x or t. With respect to x,

$$\sum_0^1 x^2 t\,dx = x_0^2 t\,dx + x_1^2 t\,dx + \cdots + x_{H-1}^2 t\,dx$$

(where $dx = 1/H$), and we shall see later that

$$\int_0^1 x^2 t\, dx = st(x_0^2 t\, dx + x_1^2 t\, dx + \cdots + x_{H-1}^2 t\, dx) = \tfrac{1}{3}t.$$

With respect to t, however,

$$\sum_0^1 x^2 t\, dt = x^2 t_0\, dt + x^2 t_1\, dt + \cdots + x^2 t_{K-1}\, dt,$$

and we shall see later that

$$\int_0^1 x^2 t\, dt = \tfrac{1}{2}x^2.$$

The next two examples evaluate the simplest definite integrals. These examples do it the hard way. A much better method will be developed in Section 4.2.

EXAMPLE 3   Given a constant $c > 0$, evaluate the integral $\int_a^b c\, dx$.

Figure 4.1.13 shows that for every positive real number Δx, the finite Riemann sum is

$$\sum_a^b c\, \Delta x = c(b - a).$$

By the Transfer Principle, the infinite Riemann sum in Figure 4.1.14 has the same value,

$$\sum_a^b c\, dx = c(b - a).$$

Taking standard parts,

$$\int_a^b c\, dx = c(b - a).$$

This is the familiar formula for the area of a rectangle.
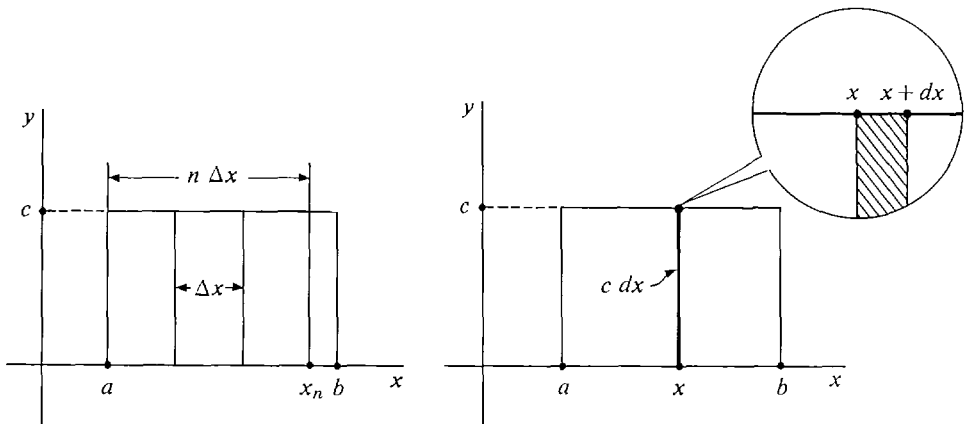
Figure 4.1.13

Figure 4.1.14

EXAMPLE 4 Given $b > 0$, evaluate the integral $\int_0^b x \, dx$.

The area under the line $y = x$ is divided into vertical strips of width dx. Study Figure 4.1.15. The area of the lower region A is the infinite Riemann sum

$$\text{area of } A = \sum_0^b x \, dx.$$

(1)

By symmetry, the upper region B has the same area as A;

(2)
$$\text{area of } A = \text{area of } B.$$

Call the remaining region C, formed by the infinitesimal squares along the diagonal. Thus

(3)
$$\text{area of } A + \text{area of } B + \text{area of } C = b^2.$$

Each square in C has height dx except the last one, which may be smaller, and the widths add up to b, so

(4)
$$0 \leq \text{area of } C \leq b \, dx.$$

Putting (1)–(4) together,

$$2 \sum_0^b x \, dx \leq b^2 \leq \left(2 \sum_0^b x \, dx \right) + b \, dx.$$

Since $b \, dx$ is infinitesimal,

$$2 \sum_0^b x \, dx \approx b^2,$$

$$\sum_0^b x \, dx \approx \frac{b^2}{2}.$$

Taking standard parts, we have

$$\int_0^b x \, dx = \frac{b^2}{2}.$$

Figure 4.1.15

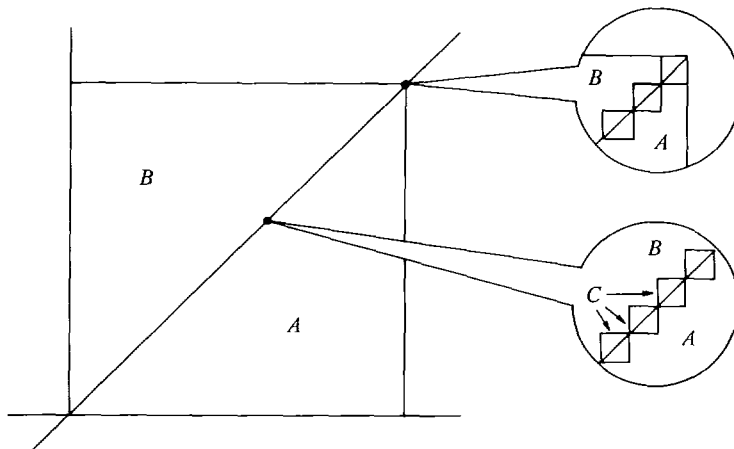## PROBLEMS FOR SECTION 4.1

Compute the following finite Riemann sums. If a hand calculator is available, the Riemann sums can also be computed with $\Delta x = \frac{1}{10}$.

1	$\sum_0^1 (3x + 1)\,\Delta x,$	$\Delta x = \frac{1}{3}$	2	$\sum_0^1 (3x + 1)\,\Delta x,$	$\Delta x = \frac{2}{5}$	
3	$\sum_{-1}^1 (3x + 1)\,\Delta x,$	$\Delta x = \frac{1}{4}$	4	$\sum_0^1 2x^2\,\Delta x,$	$\Delta x = \frac{1}{4}$	
5	$\sum_{-1}^1 2x^2\,\Delta x,$	$\Delta x = \frac{1}{4}$	6	$\sum_0^5 (2x - 1)\,\Delta x,$	$\Delta x = 1$	
7	$\sum_0^5 (2x - 1)\,\Delta x,$	$\Delta x = 2$	8	$\sum_{-1}^1 (x^2 - 1)\,\Delta x,$	$\Delta x = \frac{1}{2}$	
9	$\sum_0^2 (x^2 - 1)\,\Delta x,$	$\Delta x = \frac{1}{2}$	10	$\sum_{-1}^1 (x^2 - 1)\,\Delta x,$	$\Delta x = \frac{3}{10}$	
11	$\sum_{-4}^3 (5x^2 - 12)\,\Delta x,$	$\Delta x = 2$	12	$\sum_{-4}^3 (5x^2 - 12)\,\Delta x,$	$\Delta x = 1$	
13	$\sum_1^3 (1 + 1/x)\,\Delta x,$	$\Delta x = \frac{1}{3}$	14	$\sum_0^5 10^{-2x}\,\Delta x.$	$\Delta x = \frac{1}{2}$	
15	$\sum_{-1}^0 x^4\,\Delta x,$	$\Delta x = \frac{1}{4}$	16	$\sum_{-1}^1 2x^3\,\Delta x,$	$\Delta x = \frac{1}{2}$	
17	$\sum_0^\pi \sqrt{x}\,\Delta x,$	$\Delta x = 1$	18	$\sum_{-2}^9 |x - 4|\,\Delta x,$	$\Delta x = 2$	
19	$\sum_0^\pi \sin x\,\Delta x,$	$\Delta x = \pi/4$	20	$\sum_0^\pi \sin^2 x\,\Delta x.$	$\Delta x = \pi/4$	
21	$\sum_0^1 e^x\,\Delta x,$	$\Delta x = 1/5$	22	$\sum_0^1 xe^x\,\Delta x,$	$\Delta x = 1/5$	
23	$\sum_1^5 \ln x\,\Delta x,$	$\Delta x = 1$	24	$\sum_1^5 \frac{\ln x}{x}\,\Delta x.$	$\Delta x = 1$	

☐ 25 Let b be a positive real number and n a positive integer. Prove that if $\Delta x = b/n$,

$$\sum_0^b x\,\Delta x = (1 + 2 + \cdots + (n - 1))\,\Delta x^2.$$

Using the formula $1 + 2 + \cdots + (n - 1) = \dfrac{n(n - 1)}{2}$, prove that

$$\sum_0^b x\,\Delta x = (1 - 1/n)b^2/2.$$

☐ 26 Let H be a positive infinite hyperinteger and $dx = b/H$. Using the Transfer Principle and Problem 25, prove that $\int_0^b x\,dx = b^2/2$.

☐ 27 Let b be a positive real number, n a positive integer, and $\Delta x = b/n$. Using the formula

$$1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 = \frac{n(n - 1)(2n - 1)}{6},$$

prove that

$$\sum_0^b x^2\,\Delta x = \frac{n(n - 1)(2n - 1)}{6}\frac{b^3}{n^3}.$$

☐ 28 Use Problem 27 to show that $\int_0^b x^2\,dx = b^3/3$.

4.2 FUNDAMENTAL THEOREM OF CALCULUS

In this section we shall state five basic theorems about the integral, culminating in the Fundamental Theorem of Calculus. Right now we can only approximate a definite integral by the laborious computation of a finite Riemann sum. At the end of this section we will be in a position easily to compute exact values for many definite integrals. The key to the method is the Fundamental Theorem. Our first theorem shows that we are free to choose any positive infinitesimal we wish for dx in the definite integral.

THEOREM 1

Given a continuous function f on $[a, b]$ and two positive infinitesimals dx and du, the definite integrals with respect to dx and du are the same,

$$\int_a^b f(x)\, dx = \int_a^b f(u)\, du.$$

From now on when we write a definite integral $\int_a^b f(x)\, dx$, it is understood that dx is a positive infinitesimal. By Theorem 1, it doesn't matter which infinitesimal.

The proof of Theorem 1 is based on the following intuitive idea. Figure 4.2.1 shows the two Riemann sums $\sum_a^b f(x)\, dx$ and $\sum_a^b f(u)\, du$. We see from the figure that the difference $\sum_a^b f(x)\, dx - \sum_a^b f(u)\, du$ is a sum of rectangles of infinitesimal height. These difference rectangles all lie between the horizontal lines $y = -\varepsilon$ and $y = \varepsilon$, where ε is the largest height. Thus $-\varepsilon(b - a) \leq \sum_a^b f(x)\, dx - \sum_a^b f(u)\, du \leq \varepsilon(b - a)$. Taking standard parts,

$$0 \leq \int_a^b f(x)\, dx - \int_a^b f(u)\, du \leq 0,$$
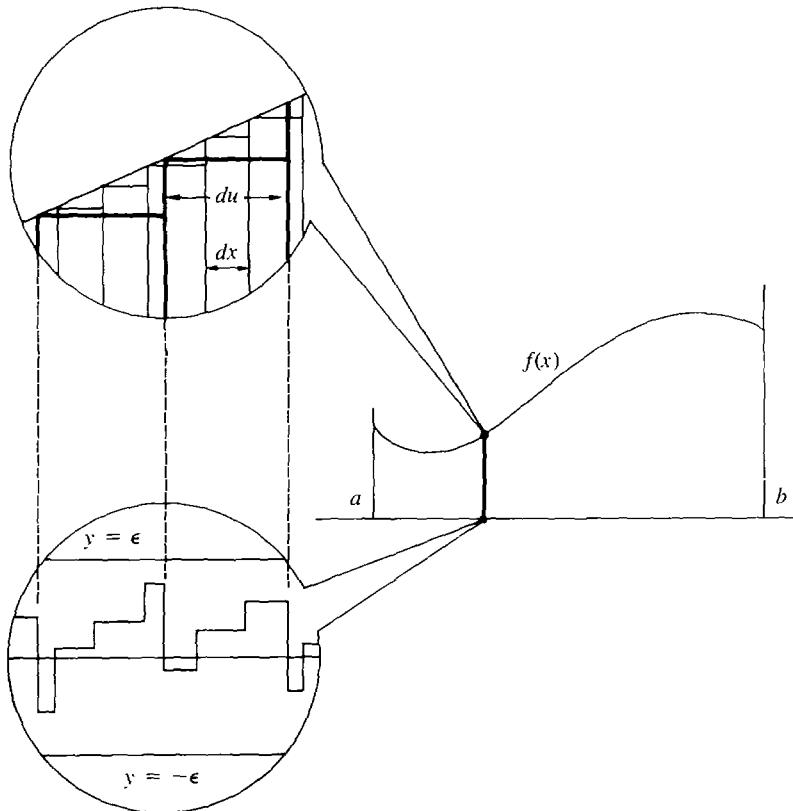
$$\int_a^b f(x)\, dx = \int_a^b f(u)\, du.$$

Figure 4.2.1

Theorem 1 shows that whenever Δx is positive infinitesimal, the Riemann sum is infinitely close to the definite integral,

$$\sum_a^b f(x)\,\Delta x \approx \int_a^b f(x)\,dx.$$

This fact can also be expressed in terms of limits. It shows that the Riemann sum approaches the definite integral as Δx approaches 0 from above, in symbols

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0^+} \sum_a^b f(x)\,\Delta x.$$

Given a continuous function f on an interval I, Theorem 1 shows that the definite integral is a real function of two variables a and b,

$$A(a, b) = \int_a^b f(x)\,dx, \qquad a, b \text{ in } I.$$

We now formally define the area as the definite integral shown in Figure 4.2.2.
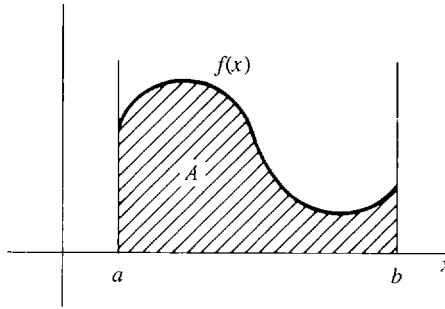
Figure 4.2.2

DEFINITION

If f is continuous and $f(x) \geq 0$ on $[a, b]$, the area of the region below the curve $y = f(x)$ from a to b is defined as the definite integral:

$$\text{Area} = \int_a^b f(x)\,dx.$$

The next two theorems give basic properties of the integral.

THEOREM 2 (The Rectangle Property)

Suppose f is continuous and has minimum value m and maximum value M on a closed interval $[a, b]$. Then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

That is, the area of the region under the curve is between the area of the rectangle whose height is the minimum value of f and the area of the rectangle whose height is the maximum value of f in the interval $[a, b]$.

The Extreme Value Theorem is needed to show that the minimum value m and maximum value M exist. The rectangle of height m is called the *inscribed rectangle* of the region, and the rectangle of height M is called the *circumscribed rectangle*. From Figure 4.2.3, we see that the inscribed rectangle is a subset of the region under the curve, which is in turn a subset of the circumscribed rectangle. The Rectangle Property says that the area of the region is between the areas of the inscribed and circumscribed rectangles.
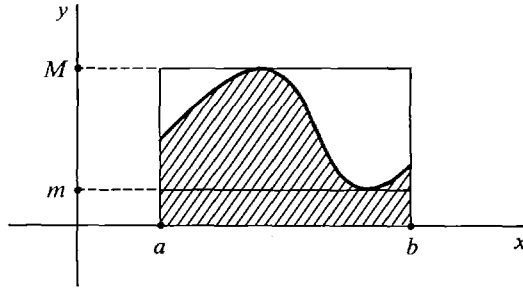
Figure 4.2.3 The Rectangle Property

PROOF By Theorem 1, any positive infinitesimal may be chosen for dx. Let us choose a positive infinite hyperinteger H and let $dx = (b - a)/H$. Then dx evenly divides $b - a$; that is, the interval $[a, b]$ is divided into H subintervals of exactly the same length dx. Then

$$\sum_{a}^{b} m \, dx = m \cdot H \cdot dx = m(b - a),$$

$$\sum_{a}^{b} M \, dx = M \cdot H \cdot dx = M(b - a).$$

For each x, we have $m \le f(x) \le M$. Adding up and taking standard parts, we obtain the required formula.

$$\sum_{a}^{b} m \, dx \le \sum_{a}^{b} f(x) \, dx \le \sum_{a}^{b} M \, dx,$$

$$m(b - a) \le \int_{a}^{b} f(x) \, dx \le M(b - a).$$

One useful consequence of the Rectangle Property is that the integral of a positive function is positive and the integral of a negative function is negative:

If $f(x) > 0$ on $[a, b]$,   then $0 < m(b - a) \le \int_{a}^{b} f(x) \, dx$.

If $f(x) < 0$ on $[a, b]$,   then $\int_{a}^{b} f(x) \, dx \le M(b - a) < 0$.

The definite integral of a negative function $f(x) = -g(x)$ from a to b is just the negative of the area of the region above the curve and below the x axis. This is because

$$f(x) \, dx = -g(x) \, dx,$$

$$\sum_a^b f(x)\, dx = -\sum_a^b g(x)\, dx,$$

$$\int_a^b f(x)\, dx = -\int_a^b g(x)\, dx.$$

(See Figure 4.2.4.)
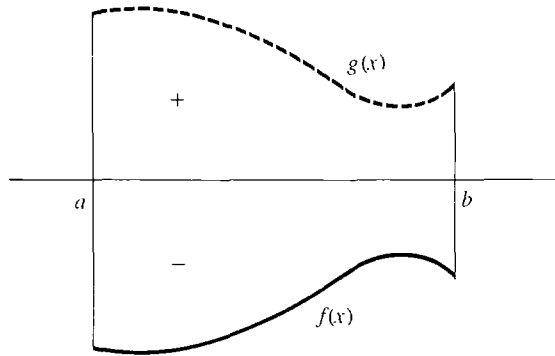
Figure 4.2.4

THEOREM 3 (The Addition Property)

Suppose f is continuous on an interval I. Then for all a, b, c in I,

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

This property is illustrated in Figure 4.2.5 for the case $a < b < c$. The Addition Property holds even if the points a, b, c are in some other order on the real line, such as $c < a < b$.

Figure 4.2.5

PROOF First suppose that $a < b < c$. Choose a dx that evenly divides the first interval length $b - a$. This simplifies our computation because it makes b a partition point, $b = a + H\, dx$. Then, as Figure 4.2.6 suggests,

$$\sum_a^c f(x)\, dx = \sum_a^b f(x)\, dx + \sum_b^c f(x)\, dx.$$

Taking standard parts we have the desired formula

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$
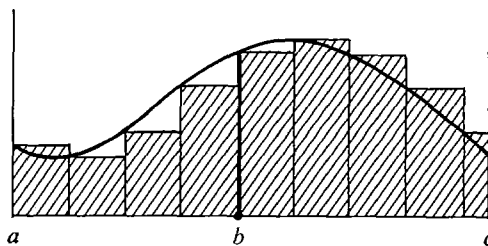
Figure 4.2.6

To illustrate the other cases, we prove the Addition Property when $c < a < b$. The previous case gives

$$\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx.$$

Since reversing the endpoints changes the sign of the integral,

$$-\int_b^c f(x)\,dx = -\int_a^c f(x)\,dx + \int_a^b f(x)\,dx,$$

and the desired formula

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

follows.

The definite integral of a curve can be thought of as area even if the curve crosses the x-axis. The curve in Figure 4.2.7 is positive from a to b and negative from b to c, crossing the x-axis at b. The integral $\int_a^b f(x)\,dx$ is a positive number and the integral $\int_b^c f(x)\,dx$ is a negative number. By the Addition Property, the integral
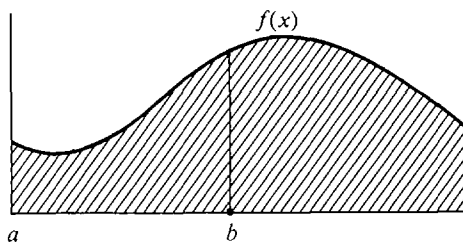
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

is equal to the area from a to b minus the area from b to c. The definite integral $\int_a^c f(x)\,dx$ always gives the net area between the x-axis and the curve, counting areas above the x-axis as positive and areas below the x-axis as negative.

The definite integral $\int_u^v f(t)\,dt$ is a real function of two variables u and v and does not depend on the dummy variable t. If we replace u by a constant a and v by the variable x, we obtain a real function of one variable x, given by

$$F(x) = \int_a^x f(t)\,dt.$$

Our fourth theorem states that this new function is continuous.
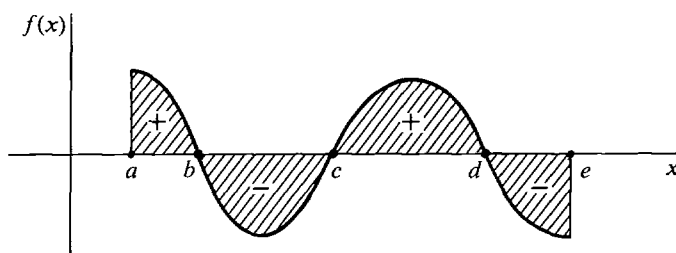
Figure 4.2.7

THEOREM 4

Let f be continuous on an interval I. Choose a point a in I. Then the function $F(x)$ defined by

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on I.

SKETCH OF PROOF   Let c be in I, and let x be infinitely close to c and between the endpoints of I. By the Addition Property,

$$\int_a^c f(t)\, dt = \int_a^x f(t)\, dt + \int_x^c f(t)\, dt,$$

$$\int_a^c f(t)\, dt - \int_a^x f(t)\, dt = \int_x^c f(t)\, dt,$$

and $$F(c) - F(x) = \int_x^c f(t)\, dt.$$

This is the area of the infinitely thin strip under the curve $y = f(t)$ between $t = x$ and $t = c$ (see Figure 4.2.8). The strip has width $\Delta x = c - x$. By the Rectangle Property, its area is between $m\,\Delta x$ and $M\,\Delta x$ and hence is infinitely small. Therefore $F(x)$ is infinitely close to $F(c)$, and F is continuous on I.
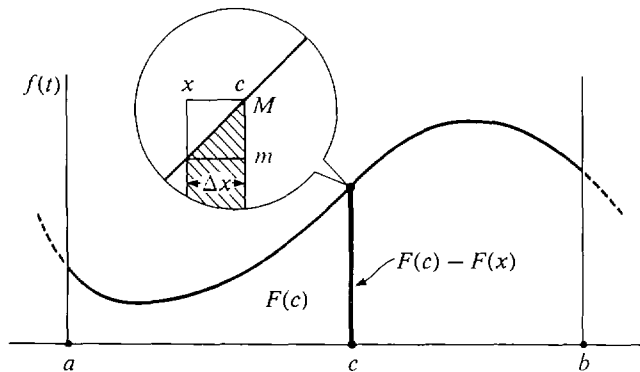
Figure 4.2.8

Our fifth theorem, the Fundamental Theorem of Calculus, shows that the definite integral can be evaluated by means of antiderivatives. The process of antidifferentiation is just the opposite of differentiation. To keep things simple, let I be an open interval, and assume that all functions mentioned have domain I.

DEFINITION

Let f and F be functions with domain I. If f is the derivative of F, then F is called an **antiderivative** of f.

For example, suppose a particle is moving upward along the y-axis with velocity $v = f(t)$ and position $y = F(t)$ at time t. The position $y = F(t)$ is an antiderivative of the velocity $v = f(t)$. We shall discuss antiderivatives in more detail in the next section. We are now ready for the Fundamental Theorem.

FUNDAMENTAL THEOREM OF CALCULUS

Suppose f is continuous on its domain, which is an open interval I.

(i) *For each point a in I, the definite integral of f from a to x considered as a function of x is an antiderivative of f. That is,*

$$d\left(\int_a^x f(t)\, dt \right) = f(x)\, dx.$$

(ii) *If F is any antiderivative of f, then for any two points (a, b) in I the definite integral of f from a to b is equal to the difference $F(b) - F(a)$,*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus is important for two reasons. First, it shows the relation between the two main notions of calculus: the derivative, which corresponds to velocity, and the integral, which corresponds to area. It shows that differentiation and integration are "inverse" processes. Second, it gives a simple method for computing many definite integrals.

EXAMPLE 1

(a) Find $\int_a^b c\, dx$. Since cx is an antiderivative of c,

$$\int_a^b c\, dx = cb - ca = c(b - a).$$

(b) Find $\int_a^b x\, dx$. $\frac{1}{2}x^2$ is an antiderivative of x. Thus

$$\int_a^b x\, dx = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2.$$

The above example gives the same result that we got before but is much simpler. We can easily go further.

EXAMPLE 2 Find $\int_a^b x^2\, dx$. $x^3/3$ is an antiderivative of x^2 because

$$\frac{d(x^3/3)}{dx} = \frac{3x^2}{3} = x^2.$$

Therefore

$$\int_a^b x^2\, dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

This gives the area of the region under the curve $y = x^2$ between a and b (Figure 4.2.9).
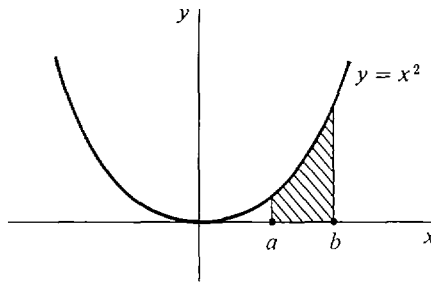
Figure 4.2.9

If a particle moves along the y-axis with continuous velocity $v = f(t)$, the position $y = F(t)$ is an antiderivative of the velocity, because $v = dy/dt$. The Fundamental Theorem of Calculus shows that the distance moved (the change in y) between times $t = a$ and $t = b$ is equal to the definite integral of the velocity,

$$\text{distance moved} = F(b) - F(a) = \int_a^b f(t)\,dt.$$

EXAMPLE 3   A particle moves along the y-axis with velocity $v = 8t^3$ cm/sec. How far does it move between times $t = -1$ and $t = 2$ sec? The function $G(t) = 2t^4$ is an antiderivative of the velocity $v = 8t^3$. Thus the definite integral is

$$\text{distance moved} = \int_{-1}^2 8t^3\,dt = 2 \cdot 2^4 - 2 \cdot (-1)^4 = 30 \text{ cm}.$$

EXAMPLE 4   Find $\int_0^4 \sqrt{t}\,dt$ (Figure 4.2.10). The function $\sqrt{t}$ is defined and continuous on the half-open interval $[0, \infty)$. But to apply the Fundamental Theorem we need a function continuous on an open interval that contains the limit points 0 and 4. We therefore define

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sqrt{t} & \text{for } t \geq 0. \end{cases}$$

This function is continuous on the whole real line. In particular it is continuous at 0 because if $t \approx 0$ then $f(t) \approx 0$. The function

$$F(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{2}{3}t^{3/2} & \text{for } t \geq 0 \end{cases}$$
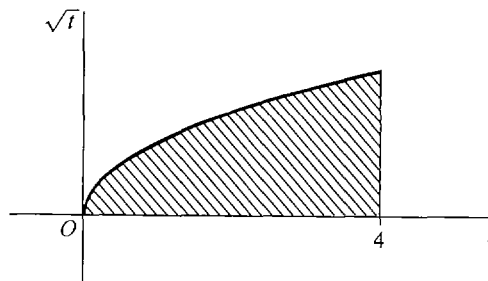
Figure 4.2.10

is an antiderivative of f. Then

$$\int_0^4 \sqrt{t}\,dt = F(4) - F(0) = (\tfrac{2}{3}\cdot 4^{3/2} - \tfrac{2}{3}\cdot 0^{3/2}) = \tfrac{16}{3}.$$

In the next section we shall develop some methods for finding antiderivatives. The antiderivative of a very simple function may turn out to be a "new" function which we have not yet given a name.

EXAMPLE 5 The only way we can show that the function $f(x) = \sqrt{1 + x^4}$ has an antiderivative is to take a definite integral

$$\int_0^x \sqrt{1 + t^4}\,dt.$$

This is a "new" function that cannot be expressed in terms of algebraic, trigonometric, and exponential functions without calculus.

The Fundamental Theorem can also be used to find the derivative of a function which is defined as a definite integral with a variable limit of integration. This can be done without actually evaluating the integral.

EXAMPLE 6 Let $y = \int_x^2 \sqrt{1 + t^2}\,dt$. Then $y = -\int_2^x \sqrt{1 + t^2}\,dt$,

and $\qquad dy = -d\left(\int_2^x \sqrt{1 + t^2}\,dt\right) = -\sqrt{1 + x^2}\,dx.$

EXAMPLE 7 Let $y = \int_3^{x^2 + x} \dfrac{1}{t^3 + 1}\,dt.$

Let $u = x^2 + x$. Then

$$\frac{du}{dx} = (2x + 1), \qquad y = \int_3^u \frac{1}{t^3 + 1}\,dt, \qquad \frac{dy}{du} = \frac{1}{u^3 + 1}.$$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u^3 + 1}(2x + 1) = \frac{2x + 1}{(x^2 + x)^3 + 1}.$$

We conclude this section with a proof of the Fundamental Theorem of Calculus.

PROOF (i) Let $F(x)$ be the area under the curve $y = f(t)$ from a to x,

$$F(x) = \int_a^x f(t)\,dt.$$

Imagine that the vertical line cutting the t-axis at x moves to the right as in Figure 4.2.11.
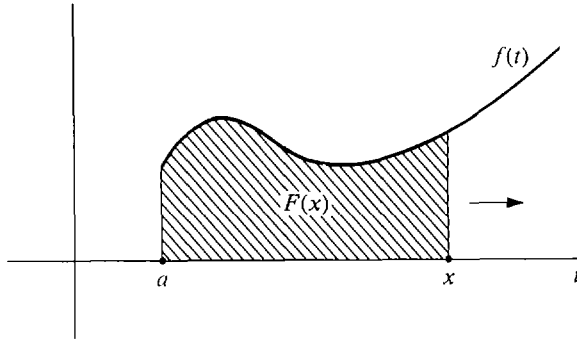
Figure 4.2.11

We show that the rate of change of $F(x)$ is equal to the length $f(x)$ of the moving vertical line.

Suppose x increases by an infinitesimal amount $\Delta x > 0$. Then

$$F(x + \Delta x) - F(x) = \int_x^{x+\Delta x} f(t)\, dt$$

is the area of an infinitely thin strip of width Δx and height infinitely close to $f(x)$. By the Rectangle Property the area of the strip is between the inscribed and circumscribed rectangles (Figure 4.2.12),

$$m\, \Delta x \le F(x + \Delta x) - F(x) \le M\, \Delta x.$$

Dividing by Δx, $m \le \dfrac{F(x + \Delta x) - F(x)}{\Delta x} \le M.$

Since f is continuous at x, the values m and M are both infinitely close to $f(x)$, and therefore

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} \approx f(x).$$

The proof is similar when $\Delta x < 0$. Hence $F'(x) = f(x)$.
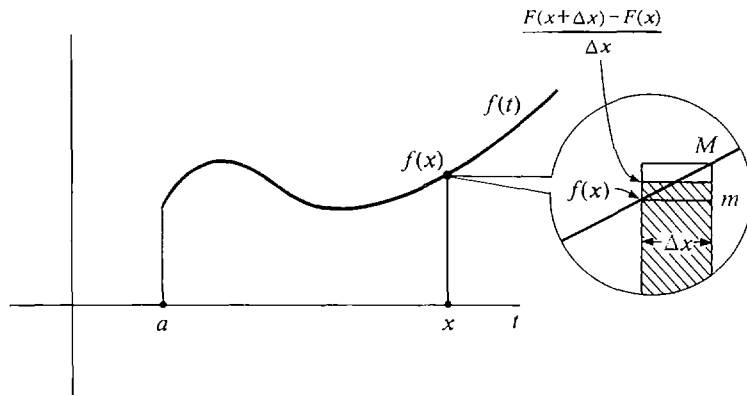
Figure 4.2.12

PROOF (ii)   Let $F(x)$ be any antiderivative of f. Then, by (i),

$$d\left(F(x) - \int_a^x f(t)\, dt\right) = f(x) - f(x) = 0.$$

In Section 3.7 on curve sketching, we saw that every function with derivative zero is constant. Thus

$$F(x) - \int_a^x f(t)\, dt = C_0, \qquad F(x) = \int_a^x f(t)\, dt + C_0$$

for some constant C_0. Then

$$F(b) - F(a) = \left(\int_a^b f(t)\, dt + C_0\right) - \left(\int_a^a f(t)\, dt + C_0\right)$$

$$= \int_a^b f(t)\, dt - 0 = \int_a^b f(t)\, dt,$$

so
$$F(b) - F(a) = \int_a^b f(x)\, dx.$$

PROBLEMS FOR SECTION 4.2

In Problems 1–14, find an antiderivative of the given function.

1	$f(x) = 8\sqrt{x}$	**2**	$f(x) = 4/\sqrt{x}$
3	$f(t) = 3t^2 + 1$	**4**	$f(x) = 5x^3$
5	$f(t) = 4 - 3t^2$	**6**	$f(z) = 2/z^2$
7	$f(s) = 7s^{-3}$	**8**	$f(t) = t^2 + t^{-2}$
9	$f(x) = (x - 6)^2$	**10**	$f(u) = (5u + 1)^2$
11	$f(y) = y^{3/2}$	**12**	$f(x) = 2/x\sqrt{x}$
13	$f(x) = |x|$	**14**	$f(t) = |2t - 4|$

15   If $F'(x) = x + x^2$ for all x, find $F(1) - F(-1)$.

16   If $F'(x) = x^4$ for all x, find $F(2) - F(1)$.

17   If $F'(t) = t^{1/3}$ for all t, find $F(8) - F(0)$.

Evaluate the definite integrals in Problems 18–22.

18 $\displaystyle\int_{-1}^{1} 2x^2\, dx$ **19** $\displaystyle\int_{-2}^{2} x^3\, dx$

20 $\displaystyle\int_{-2}^{-1} t^{-2}\, dt$ **21** $\displaystyle\int_{0}^{4} 2\sqrt{x}\, dx$

22 $\displaystyle\int_{-3}^{-2} -5x^4\, dx$

In Problems 23–27 an object moves along the y-axis. Given the velocity v, find how far the object moves between the given times t_0 and t_1.

23 $v = 2t + 5, \qquad t_0 = 0, \quad t_1 = 2$

24 $v = 4 - t, \qquad t_0 = 1, \quad t_1 = 4$

25	$v = 3,$	$t_0 = 2,$	$t_1 = 6$
26	$v = 3t^2,$	$t_0 = 1,$	$t_1 = 3$
27	$v = 10t^{-2},$	$t_0 = 1,$	$t_1 = 100$

In Problems 28–32, find the area of the region under the curve $y = f(x)$ from a to b.

28	$y = 4 - x^2,$	$a = -2,$	$b = 2$
29	$y = \sqrt{x + 2},$	$a = -2,$	$b = 2$
30	$y = 9x - x^2,$	$a = 0,$	$b = 3$
31	$y = \sqrt{x} - x,$	$a = 0,$	$b = 1$
32	$y = 3x^{1/3},$	$a = 1,$	$b = 8$

33   If $F'(t) = t - 1$ for all t and $F(0) = 2$, find $F(2)$.

34   If $F'(x) = 1 - x^2$ for all x and $F(3) = 5$, find $F(-1)$.

☐ 35   Suppose $F(x)$ and $G(x)$ have continuous derivatives and $F'(x) + G'(x) = 0$ for all x. Prove that $F(x) + G(x)$ is constant.

☐ 36   Suppose $F(x)$ and $G(x)$ have continuous derivatives such that $F'(x) \le G'(x)$ for all x. Prove that
$$F(b) - F(a) \le G(b) - G(a)$$
where $a < b$.

☐ 37   Prove that a function $F(x)$ has a constant derivative if and only if $F(x)$ is linear, i.e., of the form $F(x) = ax + b$.

☐ 38   Prove that a function $F(x)$ has a constant second derivative if and only if $F(x)$ has the form $F(x) = ax^2 + bx + c$.

☐ 39   Suppose that $F''(x) = G''(x)$ for all x. Prove that $F(x)$ and $G(x)$ differ by a linear function, that is, $G(x) = F(x) + ax + b$ for some real numbers a and b.

4.3 INDEFINITE INTEGRALS

The Fundamental Theorem of Calculus shows that every continuous function f has at least one antiderivative, namely $F(x) = \int_a^x f(t)\,dt$. Actually, f has infinitely many antiderivatives, but any two antiderivatives of f differ only by a constant. This is an important fact about antiderivatives, which we state as a theorem.

THEOREM 1

Let f be a real function whose domain is an open interval I.

(i)   *If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ is an antiderivative of $f(x)$ for every real number C.*

(ii) *If $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then $F(x) - G(x)$ is constant for all x in I. That is,*

$$G(x) = F(x) + C$$

for some real number C.

Discussion   Parts (i) and (ii) together show that if we can find one antiderivative $F(x)$ of $f(x)$, then the family of functions

$$F(x) + C, \qquad C = \text{a real number}$$

gives *all* antiderivatives of $f(x)$. We see from Figure 4.3.1 that the graph of $F(x) + C$ is just the graph of $F(x)$ moved vertically by a distance C. The graphs of $F(x)$ and $F(x) + C$ have the same slopes at every point x. For example, let $f(x) = 3x^2$. Then $F(x) = x^3$ is an antiderivative of $3x^2$ because

$$\frac{d(x^3)}{dx} = 3x^2.$$

But $x^3 + 6$ and $x^3 - \sqrt{2}$ are also antiderivatives of $3x^2$. In fact, $x^3 + C$ is an antiderivative of $3x^2$ for each real number C. Theorem 1 shows that $3x^2$ has no other antiderivatives.
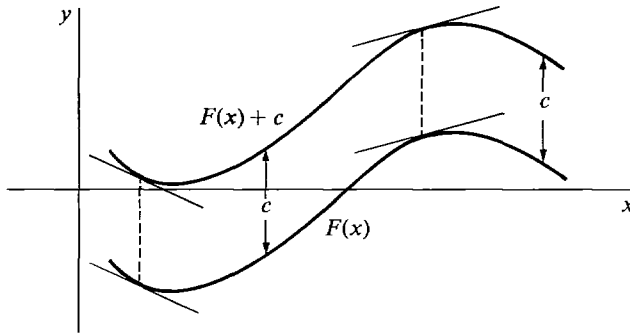
Figure 4.3.1

PROOF We prove (i) by differentiating,

$$\frac{d(F(x) + C)}{dx} = \frac{d(F(x))}{dx} + \frac{dC}{dx} = f(x) + 0 = f(x).$$

Part (ii) follows from a theorem in Section 3.7 on curve sketching. If a function has derivative zero on I, then the function is constant on I. The difference $F(x) - G(x)$ has derivative $f(x) - f(x) = 0$ and is therefore constant. We used this fact in the proof of the Fundamental Theorem of Calculus.

In computing integrals of f, we usually work with the family of all anti-derivatives of f. We shall call this whole family of functions the *indefinite integral* of f. The symbol for the indefinite integral is $\int f(x)\,dx$. If $F(x)$ is one antiderivative of f, the indefinite integral is the set of all functions of the form $F(x) + C_0$, C_0 constant. We express this with the equation

$$\int f(x)\,dx = F(x) + C.$$

It is an equation between two families of functions rather than between two single functions. C is called the *constant of integration*. To illustrate the notation,

$$\int 3x^2\,dx = x^3 + C.$$

We repeat the above definitions in concise form.

DEFINITION

*Let the domain of f be an open interval I and suppose f has an antiderivative. The family of all antiderivatives of f is called the **indefinite integral** of f and is denoted by $\int f(x)\,dx$.*

Given a function F, the family of all functions which differ from F only by a constant is written $F(x) + C$. Thus if F is an antiderivative of f we write

$$\int f(x)\,dx = F(x) + C.$$

When working with indefinite integrals, it is convenient to use differentials and dependent variables. If we introduce the dependent variable u by $u = F(x)$, then

$$du = F'(x)\,dx = f(x)\,dx.$$

Thus the equation

$$\int f(x)\,dx = F(x) + C$$

can be written in the form

$$\int du = u + C.$$

The differential symbol d and the indefinite integral symbol $\int$ behave as inverses to each other. We can start with the family of functions $u + C$, form du, and then form $\int du = u + C$ to get back where we started. Some of the rules for differentiation given in Chapter 2 can be turned around to give a set of rules for indefinite integration.

THEOREM 2

Let u and v be functions of x whose domains are an open interval I and suppose du and dv exist for every x in I.

(i) $\displaystyle\int du = u + C.$

(ii)   Constant Rule $\displaystyle\int c\,du = c\int du.$

(iii)   Sum Rule $\displaystyle\int du + dv = \int du + \int dv.$

(iv)   Power Rule $\displaystyle\int u^r\,du = \frac{u^{r+1}}{r+1} + C,$ *where r is rational, $r \neq -1$,*

and $u > 0$ *on I.*

(v) $\displaystyle\int \sin u\,du = -\cos u + C.$

(vi) $\displaystyle\int \cos u\,du = \sin u + C.$

(vii) $\displaystyle\int e^u\,du = e^u + C.$

(viii) $\displaystyle\int \frac{1}{u}\,du = \ln |u| + C \qquad (u \neq 0)$.

Discussion The Power Rule gives the integral of u^r when $r \neq -1$, while Rule (viii) gives the integral of u^r when $r = -1$. When we put $u = f(x)$ and $v = g(x)$, the Constant and Sum Rules take the form

Constant Rule $\displaystyle\int cf(x)\,dx = c \int f(x)\,dx.$

Sum Rule $\displaystyle\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$

In the Constant and Sum Rules we are multiplying a family of functions by a constant and adding two families of functions. If we do either of these two things to families of functions differing only by a constant, we get another family of functions differing only by a constant. For example,

$$7(3x^4 + C) = 21x^4 + 7C = 21x^4 + C'$$

is the family of all functions equal to $21x^4$ plus a constant. Similarly,

$$(3\sqrt{x} + C) + (5x - \sqrt{x} + D) = 5x + 2\sqrt{x} + (C + D) = 5x + 2\sqrt{x} + C'$$

is the family of all functions equal to $5x + 2\sqrt{x}$ plus a constant.

PROOF OF THEOREM 2

(i) This is just a short form of the theorem that $u + C$ is the family of all functions which have the same derivative as u.

(ii) We have $c\,du = d(cu)$, whence

$$\int c\,du = \int d(cu) = cu + C = c(u + C') = c \int du.$$

(iii) $du + dv = d(u + v)$,

$$\int du + dv = \int d(u + v) = u + v + C = \int du + \int dv.$$

(iv) $\displaystyle d\left(\frac{u^{r+1}}{r+1}\right) = \frac{(r+1)u^r}{r+1}\,du = u^r\,du,$

$$\int u^r\,du = \frac{u^{r+1}}{r+1} + C.$$

Rules (v)–(viii) are similar. Only the last formula, (viii), requires an explanation. The absolute value in $\ln |u|$ comes about by combining the two cases $u > 0$ and $u < 0$. When $u > 0$, $u = |u|$ and

$$d(\ln |u|) = d(\ln u) = \frac{1}{u}\,du.$$

When $u < 0$, $\ln u$ is undefined, but $|u| = -u$ and $\ln |u| = \ln (-u)$. Thus

$$d(\ln |u|) = d(\ln (-u)) = -\frac{1}{u}\,d(-u) = \frac{1}{u}\,du.$$

Thus, in both cases, when $u \neq 0$,

$$d(\ln|u|) = \frac{1}{u} du,$$

$$\int \frac{1}{u} du = \ln|u| + C.$$

EXAMPLE 1 $\int (2x^{-1} + 3 \sin x)\, dx = 2 \ln|x| - 3 \cos x + C.$

We can use the rules to write down at once the indefinite integral of any polynomial.

EXAMPLE 2 $\int (4x^3 - 6x^2 + 2x + 1)\, dx = x^4 - 2x^3 + x^2 + x + C.$

EXAMPLE 3 $\int \left(\frac{3}{x^2} + \sqrt{x} \right) dx = -\frac{3}{x} + \frac{2}{3} x^{3/2} + C.$

Indefinite integration is much harder than differentiation, because there are no rules for integrating the product or quotient of two functions. It often requires guesswork. The short list of rules in Theorem 1 will help, and as this course proceeds we shall add many more techniques for finding indefinite integrals.

EXAMPLE 4   Show that $\displaystyle \int \frac{dx}{(1 + x)^{1/2}(1 - x)^{3/2}} = \sqrt{\frac{1 + x}{1 - x}} + C.$

Our rules give no hint on finding this integral. However, once the answer is given to us we can easily prove that it is correct by differentiating,

$$\frac{d\sqrt{\frac{1 + x}{1 - x}}}{dx} = \frac{d((1 + x)^{1/2}(1 - x)^{-1/2})}{dx}$$

$$= (1 + x)^{1/2}(-1)(-\tfrac{1}{2})(1 - x)^{-3/2} + (1 - x)^{-1/2}(\tfrac{1}{2})(1 + x)^{-1/2}$$

$$= (1 + x)^{-1/2}(1 - x)^{-3/2}[\tfrac{1}{2}(1 + x) + \tfrac{1}{2}(1 - x)]$$

$$= \frac{1}{(1 + x)^{1/2}(1 - x)^{3/2}}.$$

Here is a warning that may prevent some common mistakes.

Warning: The integral of the product of two functions is *not* equal to the product of the integrals. The same goes for quotients. That is,

Wrong: $\displaystyle \int (uv)\, dx = \left(\int u\, dx \right)\left(\int v\, dx \right).$

For example,

Wrong: $\quad \int x(x+1)\,dx = \left(\int x\,dx\right)\left(\int (x+1)\,dx\right) = \dfrac{x^2}{2}\left(\dfrac{x^2}{2} + x\right) + C$

$$= \dfrac{x^4}{4} + \dfrac{x^3}{2} + C.$$

Correct: $\quad \int x(x+1)\,dx = \int (x^2 + x)\,dx = \dfrac{x^3}{3} + \dfrac{x^2}{2} + C.$

Wrong: $\quad \displaystyle\int \dfrac{u}{v}\,dx = \dfrac{\displaystyle\int u\,dx}{\displaystyle\int v\,dx}.$

For example,

Wrong: $\quad \displaystyle\int \dfrac{x+1}{\sqrt{x}}\,dx = \dfrac{\displaystyle\int (x+1)\,dx}{\displaystyle\int \sqrt{x}\,dx} = \dfrac{(\frac{1}{2})x^2 + x}{(\frac{2}{3})x^{3/2}} + C$

$$= \dfrac{3\sqrt{x}}{4} + \dfrac{3}{2\sqrt{x}} + C.$$

Correct: $\quad \displaystyle\int \dfrac{x+1}{\sqrt{x}}\,dx = \int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)dx = \frac{2}{3}x^{3/2} + 2\sqrt{x} + C.$

The indefinite integral can be used to solve problems of the following type. Given that a particle moves along the y-axis with velocity $v = f(t)$, and that at a certain time $t = t_0$ its position is $y = y_0$. Find the position y as a function of t.

EXAMPLE 5 A particle moves with velocity $v = 1/t^2$, $t > 0$. At time $t = 2$ it is at position $y = 1$. Find the position y as a function of t. We compute

$$\int v\,dt = \int \dfrac{1}{t^2}\,dt = -\dfrac{1}{t} + C.$$

Since $dy/dt = v$, y is one of the functions in the family $-1/t + C$. We can find the constant C by setting $t = 2$ and $y = 1$,

$$y = -\dfrac{1}{t} + C, \qquad 1 = -\dfrac{1}{2} + C, \qquad C = 1\tfrac{1}{2}.$$

Then the answer is

$$y = -\dfrac{1}{t} + 1\tfrac{1}{2}.$$

The next theorem shows that in such a problem we can always find the answer if we are given the position of the particle at just one point of time.

THEOREM 3

Suppose the domain of f is an open interval I and f has an antiderivative. Let $P(x_0, y_0)$ be any point with x_0 in I. Then f has exactly one antiderivative whose graph passes through P.

PROOF Let F be any antiderivative of f. Then $F(x) + C$ is the family of all anti-derivatives. We show that there is exactly one value of C such that the function $F(x) + C$ passes through $P(x_0, y_0)$ (Figure 4.3.2). We note that all of the following statements are equivalent:

(1) $F(x) + C$ passes through $P(x_0, y_0)$.
(2) $F(x_0) + C = y_0$.
(3) $C = y_0 - F(x_0)$.

Thus $y_0 - F(x_0)$ is the unique value of C which works.
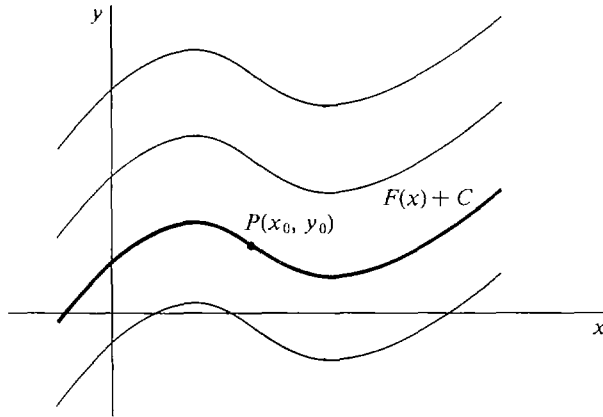
Figure 4.3.2

The Fundamental Theorem of Calculus, part (ii), may be expressed briefly as follows, where f is continuous on I.

If $\int f(x)\,dx = F(x) + C$, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

For evaluating definite integrals we introduce the convenient notation

$$F(x)\Big]_a^b = F(b) - F(a).$$

It is read "$F(x)$ evaluated from a to b."

The Constant and Sum Rules hold for definite as well as indefinite integrals:

Constant Rule $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

Sum Rule $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

The Constant Rule is shown by the computation

$$\int_a^b cf(x)\,dx = cF(b) - cF(a) = c(F(b) - F(a)) = c\int_a^b f(x)\,dx.$$

The Sum Rule is similar.

EXAMPLE 6 Evaluate the definite integral of $y = (1 + t)/t^3$ from $t = 1$ to $t = 2$ (see Figure 4.3.3).

$$\int_1^2 \frac{1+t}{t^3}\,dt = \int_1^2 (t^{-3} + t^{-2})\,dt$$

$$= \int_1^2 t^{-3}\,dt + \int_1^2 t^{-2}\,dt = \frac{t^{-2}}{-2}\Big]_1^2 + \frac{t^{-1}}{-1}\Big]_1^2$$

$$= \left(\frac{1}{(-2)\cdot 4} - \frac{1}{(-2)\cdot 1}\right) + \left(\frac{1}{-2} - \frac{1}{-1}\right) = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}.$$

Thus the area under the curve $y = (1 + t)/t^3$ from $t = 1$ to $t = 2$ is $\frac{7}{8}$.
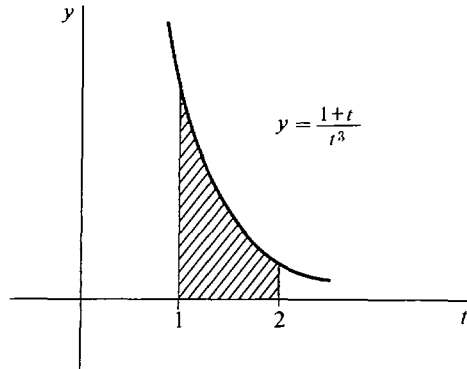
Figure 4.3.3

EXAMPLE 7 Find the area of the region under one arch of the curve $y = \sin x$ (see Figure 4.3.4).

One arch of the sine curve is between $x = 0$ and $x = \pi$. The area is the definite integral

$$\int_0^\pi \sin x\,dx = -\cos x\Big]_0^\pi$$

$$= -\cos \pi - (-\cos 0) = -(-1) - (-1) = 2.$$

The area is exactly 2.
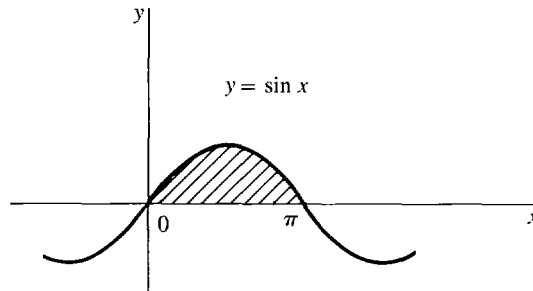
Figure 4.3.4

EXAMPLE 8   Find the area under the curve $y = -2x^{-1}$ from $x = -5$ to $x = -1$. (See Figure 4.3.5.)

The area is given by the definite integral

$$\int_{-5}^{-1} -2x^{-1}\, dx.$$

First compute the indefinite integral

$$\int -2x^{-1}\, dx = -2 \int x^{-1}\, dx = -2 \ln |x| + C.$$

Now compute the definite integral.

$$\int_{-5}^{-1} -2x^{-1}\, dx = -2 \ln |x| \Big]_{-5}^{-1}$$

$$= -2(\ln |-1| - \ln |-5|) = -2(\ln 1 - \ln 5)$$

$$= 2 \ln 5 \sim 3.219.$$
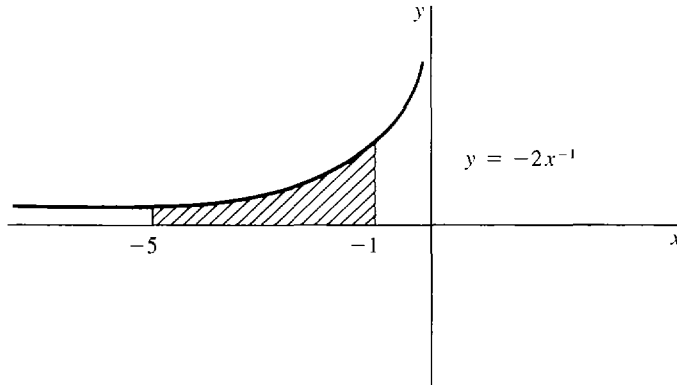
Figure 4.3.5

This example illustrates the need for the absolute value in the integration rule

$$\int x^{-1}\, dx = \ln |x| + C.$$

The natural logarithm $\ln x$ is undefined at $x = -5$ and $x = -1$, but $\ln |x|$ is defined for all $x \neq 0$. The absolute value sign is put in when integrating x^{-1} and removed when differentiating $\ln |x|$.

EXAMPLE 9   In computing definite integrals one must first make sure that the function to be integrated is continuous on the interval. For instance,

$$Incorrect: \qquad \int_{-1}^{1} \frac{1}{x^2}\, dx = \frac{x^{-1}}{-1} \bigg]_{-1}^{1} = -1 - (-(-1)) = -2.$$

This is clearly wrong because $1/x^2 > 0$ so the area under the curve cannot be negative. The mistake is that $1/x^2$ is undefined at $x = 0$ and hence the function is discontinuous at $x = 0$. Therefore the area under the curve and the definite integral

$$\int_{-1}^{1} \frac{1}{x^2} \, dx$$

are undefined (Figure 4.3.6).

$$f(x) = \frac{1}{x^2}$$

$(-1, 1)$

$(1, -1)$

$$F(x) = \frac{x^{-1}}{-1}$$
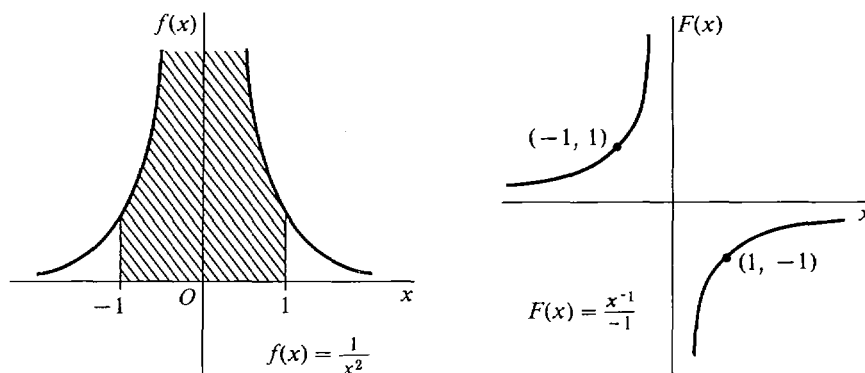
Figure 4.3.6

PROBLEMS FOR SECTION 4.3

Evaluate the following integrals.

1 $\int (1 + 2x + 3x^2) \, dx$ **2** $\int (2x^2 - 6x + 9) \, dx$

3 $\int (12t^7 - 3t^5 + 2t^2 + 1) \, dt$ **4** $\int (5 + y^{-2} - 4y^{-3}) \, dy$

5 $\int (t^{1/2} + t^{-1/2}) \, dt$ **6** $\int (2y^{1/3} - 3y^{2/3}) \, dy$

7 $\int (2x - 3)^2 \, dx$ **8** $\int (x - 2)(2x + 1) \, dx$

9 $\int (z + 1/z)^2 \, dz$ **10** $\int (z - 1/z)^2 \, dz$

11 $\int 5 \cos x \, dx$ **12** $\int (\sin x + \cos x) \, dx$

13 $\int \frac{x + 1}{x} \, dx$ **14** $\int \frac{2x^2 - 3x + 6}{x^2} \, dx$

15 $\int (1 + x^{-1})^2 \, dx$ **16** $\int 3e^x \, dx$

17 $\int (3 + \sqrt{t})(4 - 2\sqrt{t}) \, dt$ **18** $\int \frac{3s + 1}{3\sqrt{s}} \, ds$

19 $\int \frac{4 + 3\sqrt{y} + y\sqrt{y}}{y^2} \, dy$ **20** $\int (3 - x^2)(1 + 4x^2) \, dx$

21 $\int (ax^2 + bx + c) \, dx$ **22** $\int (a_3 x^3 + a_2 x^2 + a_1 x + a_0) \, dx$

23 $\displaystyle\int_{-2}^{2} (2x - 4x^3 + x^5)\, dx$ **24** $\displaystyle\int_{0}^{1} (1 + x^2 + 3x^4)\, dx$

25 $\displaystyle\int_{-1}^{1} (1 + x^2 + 3x^4)\, dx$ **26** $\displaystyle\int_{-1}^{1} e^x\, dx$

27 $\displaystyle\int_{0}^{\pi} \cos x\, dx$ **28** $\displaystyle\int_{0}^{\pi/2} \cos x\, dx$

29 $\displaystyle\int_{1}^{2} 3x^{-1}\, dx$ **30** $\displaystyle\int_{2}^{5} \frac{x-1}{x}\, dx$

31 $\displaystyle\int_{-3}^{-1} \frac{1}{x}\, dx$

In Problems 32–36, find the position y as a function of t given the velocity $v = dy/dt$ and the value of y at one point of time.

32 $v = 2t + 3,$ $y = 0$ when $t = 0$

33 $v = 4t^2 - 1,$ $y = 2$ when $t = 0$

34 $v = 3t^4,$ $y = 0$ when $t = -1$

35 $v = 2 \sin t,$ $y = 10$ when $t = 0$

36 $v = 3t^{-1},$ $y = 1$ when $t = 1$

In Problems 37–42, find the position y and velocity v as a function of t given the acceleration a and the values of y and v at $t = 0$ or $t = 1$.

37 $a = t,$ $v = 0$ and $y = 1$ when $t = 0$

38 $a = -32,$ $v = 10$ and $y = 0$ when $t = 0$

39 $a = 3t^2,$ $v = 1$ and $y = 2$ when $t = 0$

40 $a = 1 - \sqrt{t},$ $v = -2$ and $y = 1$ when $t = 0$

41 $a = t^{-3},$ $v = 1$ and $y = 0$ when $t = 1$

42 $a = -\sin t,$ $v = 0$ and $y = 4$ when $t = 0$

43   Which of the following definite integrals are undefined?

(a) $\displaystyle\int_{-1}^{2} \frac{1}{x}\, dx$ (b) $\displaystyle\int_{1}^{2} \frac{1}{x}\, dx$

(c) $\displaystyle\int_{-1}^{0} \sqrt{-y}\, dy$ (d) $\displaystyle\int_{-1}^{0} \sqrt{y}\, dy$

(e) $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ (f) $\displaystyle\int_{-2}^{2} \sqrt{x^2 - 4}\, dx$

(g) $\displaystyle\int_{-1}^{1} \frac{1}{u^2 - 1}\, du$ (h) $\displaystyle\int_{-2}^{-1} \sqrt{t^2 - 1}\, dt$

(i) $\displaystyle\int_{-2}^{2} \sqrt{t^2 - 1}\, dt$ (j) $\displaystyle\int_{-3}^{3} |x - 1|\, dx$

(k) $\displaystyle\int_{-1}^{1} \tan x\, dx$ (l) $\displaystyle\int_{0}^{\pi} \tan x\, dx$

44   Find the function f such that f' is constant, $f(0) = f'(0)$ and $f(2) = f'(2)$.

45   An object moves with acceleration $a = 6t$. Find its position y as a function of t, given that $y = 1$ when $t = 0$ and $y = 4$ when $t = 1$.

46   Find the function h such that h'' is constant, $h(1) = 1$, $h(2) = 2$, and $h(3) = 3$.

☐ **47**   Suppose that $F''(x)$ exists for all x, and let (x_0, y_0) and (x_1, y_1) be two given points. Prove that there is exactly one function $G(x)$ such that

$$G(x_0) = y_0$$
$$G'(x_1) = y_1$$
$$G''(x) = F''(x) \qquad \text{for all } x.$$

☐ **48** Assume that $F''(x)$ exists for all x, and let (x_1, y_1) and (x_2, y_2) be two points with $x_1 \neq x_2$. Prove that there is exactly one function $G(x)$ such that $G''(x) = F''(x)$ for all x, and the graph of G passes through the two points (x_1, y_1) and (x_2, y_2).

4.4 INTEGRATION BY CHANGE OF VARIABLES

We have seen that the sum, constant, and power rules for differentiation can be turned around to give the sum, constant, and power rules for integration. In this section we shall show how to make use of the Chain Rule for differentiation in problems of integration. The Chain Rule will lead to the important method of *integration by change of variables*. The basic idea is to try to simplify the function to be integrated by changing from one independent variable to another.

If F is an antiderivative of f and we take u as the independent variable, then $\int f(u) \, du$ is a family of functions of u,

$$\int f(u) \, du = F(u) + C.$$

But if we take x as the independent variable and introduce u as a dependent variable $u = g(x)$, then du and $\int f(u) \, du$ mean the following:

$$du = g'(x) \, dx, \qquad \int f(u) \, du = \int f(g(x))g'(x) \, dx = H(x) + C.$$

The notation $\int f(u) \, du$ always stands for a family of functions of the independent variable, which in some cases is another variable such as x. The next theorem can be used as follows. To integrate a given function of x, properly choose a new variable $u = g(x)$ and integrate a new function with respect to u.

DEFINITION

> Let I and J be intervals. We say that a function g **maps J into I** if for every point x in J, $g(x)$ is defined and belongs to I (Figure 4.4.1).
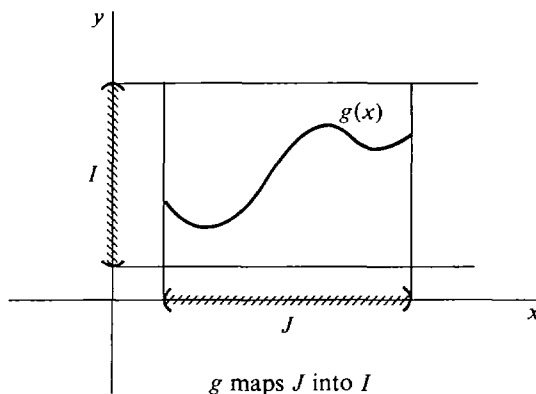
Figure 4.4.1

g maps J into I

THEOREM 1 (Indefinite Integration by Change of Variables)

Suppose I and J are open intervals, f has domain I, g maps J into I, and g is differentiable on J. Assume that when we take u as the independent variable,

$$\int f(u)\,du = F(u) + C.$$

Then when x is the independent variable and u = g(x),

$$\int f(u)\,du = F(g(x)) + C.$$

PROOF   Let $H(x) = F(g(x))$. For any x in J, the derivatives $g'(x)$ and $F'(g(x)) = f(g(x))$ exist. Therefore by the Chain Rule,

$$H'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

It follows that

$$\int f(g(x))g'(x)\,dx = H(x) + C = F(g(x)) + C.$$

So when $u = g(x)$, we have

$$f(u)\,du = f(g(x))g'(x)\,dx, \qquad \int f(u)\,du = F(g(x)) + C.$$

Theorem 1 gives another proof of the general power rule

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1,$$

where u is given as a function of the independent variable x, from the simpler power rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1,$$

where x is the independent variable.

EXAMPLE 1   Find $\int (4x+1)^3 + (4x+1)^2 + (4x+1)\,dx$. Let $u = 4x + 1$. Then $du = 4\,dx$, $dx = \frac{1}{4}\,du$. Hence

$$\int (4x+1)^3 + (4x+1)^2 + (4x+1)\,dx$$

$$= \int (u^3 + u^2 + u) \cdot \frac{1}{4}\,du = \frac{1}{4}\left(\frac{u^4}{4} + \frac{u^3}{3} + \frac{u^2}{2}\right) + C$$

$$= \frac{1}{4}\left[\frac{(4x+1)^4}{4} + \frac{(4x+1)^3}{3} + \frac{(4x+1)^2}{2}\right] + C.$$

EXAMPLE 2   Find $\int \dfrac{-1}{x^2(1 + 1/x)^2}\,dx$.

Let $u = 1 + 1/x$. Then $du = -1/x^2\,dx$ and thus

$$\frac{-1}{x^2(1 + 1/x)^2} \, dx = \frac{1}{u^2} \, du.$$

So $$\int \frac{-1}{x^2(1 + 1/x)^2} \, dx = \int \frac{1}{u^2} \, du = \frac{u^{-1}}{-1} + C = -\frac{1}{1 + 1/x} + C.$$

In a simple problem such as this example, we can save writing by using the term $1 + 1/x$ instead of introducing a new letter u,

$$\int \frac{-1}{x^2(1 + 1/x)^2} \, dx = \int \frac{1}{(1 + 1/x)^2} \, d\left(1 + \frac{1}{x}\right) = \frac{(1 + 1/x)^{-1}}{-1} + C.$$

In examples such as the above one, the trick is to find a new variable u such that the expression becomes simpler when we change variables. This usually must be done by an "educated" trial and error process.

One must be careful to express dx in terms of du before integrating with respect to u.

EXAMPLE 3 Find $\int(1 + 5x)^2 \, dx$. Let $u = 1 + 5x$. For emphasis we shall do it correctly and incorrectly.

Correct: $$du = 5 \, dx, \qquad dx = \tfrac{1}{5} \, du,$$

$$\int (1 + 5x)^2 \, dx = \int u^2 \cdot \frac{1}{5} \, du = \frac{u^3}{15} + C = \frac{(1 + 5x)^3}{15} + C.$$

Incorrect: $$\int (1 + 5x)^2 \, dx = \int u^2 \, dx = \frac{u^3}{3} + C = \frac{(1 + 5x)^3}{3} + C.$$

Incorrect: $$\int (1 + 5x)^2 \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{(1 + 5x)^3}{3} + C.$$

EXAMPLE 4 Find $\int x^3 \sqrt{2 - x^2} \, dx$. Let $u = 2 - x^2$, $du = -2x \, dx$, $dx = du/(-2x)$. We try to express the integral in terms of u.

$$\int x^3 \sqrt{2 - x^2} \, dx = \int x^3 \sqrt{u} \, \frac{du}{-2x} = \int -\frac{1}{2} x^2 \sqrt{u} \, du.$$

Since $u = 2 - x^2$, $x^2 = 2 - u$. Therefore

$$\int -\tfrac{1}{2} x^2 \sqrt{u} \, du = \int -\tfrac{1}{2}(2 - u)\sqrt{u} \, du = \int -\sqrt{u} + \tfrac{1}{2} u^{3/2} \, du$$

$$= -\tfrac{2}{3} u^{3/2} + \tfrac{1}{2} \cdot \tfrac{2}{5} u^{5/2} + C$$

$$= -\tfrac{2}{3}(2 - x^2)^{3/2} + \tfrac{1}{5}(2 - x^2)^{5/2} + C.$$

We next describe the method of *definite integration by change of variables*. In a definite integral

$$\int_a^b h(x) \, dx$$

it is always understood that x is the independent variable and we are integrating between the limits $x = a$ and $x = b$. Thus when we change to a new independent

variable u, we must also change the limits of integration. The theorem below will show that if $u = c$ when $x = a$ and $u = d$ when $x = b$, then c and d will be the new limits of integration.

THEOREM 2 (Definite Integration by Change of Variables)

Suppose I and J are open intervals, f is continuous and has an antiderivative on I, g has a continuous derivative on J, and g maps J into I. Then for any two points a and b in J,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

PROOF Let F be an antiderivative of f. Then by Theorem 1, $H(x) = F(g(x))$ is an antiderivative of $h(x) = f(g(x))g'(x)$. Since f, g, and g' are continuous, h is continuous on J. Then by the Fundamental Theorem of Calculus,

$$\int_a^b f(g(x))g'(x)\,dx = H(b) - H(a) = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 5   Find the area under the line $y = 1 + 3x$ from $x = 0$ to $x = 1$. This can be done either with or without a change of variables.

Without change of variable: $\int(1 + 3x)\,dx = x + 3x^2/2 + C$, so

$$\int_0^1 (1 + 3x)\,dx = x + \frac{3x^2}{2}\Bigg]_0^1 = \left(1 + \frac{3 \cdot 1^2}{2}\right) - \left(0 + \frac{3 \cdot 0^2}{2}\right) = \frac{5}{2}.$$

With change of variable: Let $u = 1 + 3x$. Then $du = 3\,dx$, $dx = \frac{1}{3}\,du$. When $x = 0$, $u = 1 + 3 \cdot 0 = 1$. When $x = 1$, $u = 1 + 3 \cdot 1 = 4$.

$$\int_0^1 (1 + 3x)\,dx = \int_1^4 u \cdot \frac{1}{3}\,du = \frac{u^2}{6}\Bigg]_1^4 = \frac{16}{6} - \frac{1}{6} = \frac{15}{6} = \frac{5}{2}.$$

Example 5 shows us that $\int_0^1 (1 + 3x)\,dx = \int_1^4 (u/3)\,du$; that is, the areas shown in Figure 4.4.2 are the same.
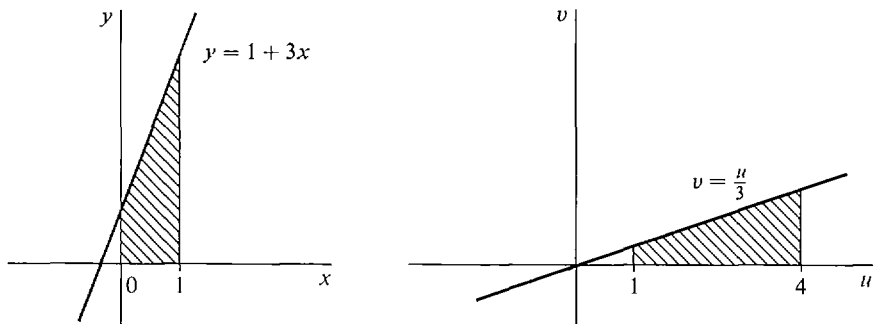
Figure 4.4.2
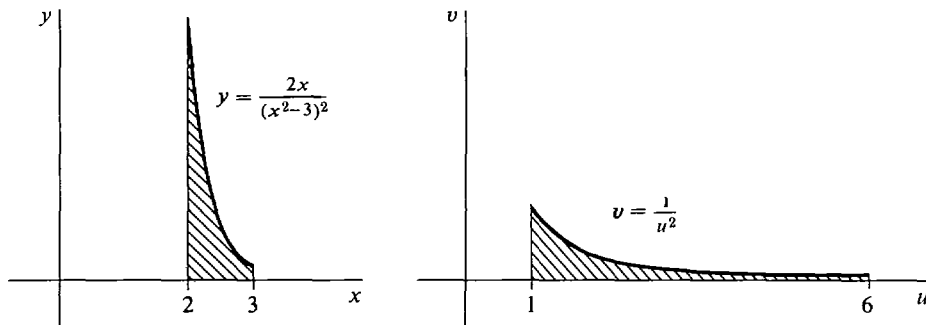
Figure 4.4.3

EXAMPLE 6   Find the area under the curve $y = 2x/(x^2 - 3)^2$ from $x = 2$ to $x = 3$ (Figure 4.4.3).

Let $u = x^2 - 3$. Then $du = 2x \, dx$. At $x = 2$, $u = 2^2 - 3 = 1$. At $x = 3$, $u = 3^2 - 3 = 6$. Then

$$\int_2^3 \frac{2x}{(x^2 - 3)^2} \, dx = \int_1^6 \frac{1}{u^2} \, du = -\frac{1}{u} \bigg]_1^6 = 1 - \frac{1}{6} = \frac{5}{6}.$$

EXAMPLE 7   Find $\int_0^1 \sqrt{1 - x^2} \, x \, dx$. The function $\sqrt{1 - x^2} \, x$ as given is only defined on the closed interval $[-1, 1]$. In order to use Theorem 2, we extend it to the open interval $J = (-\infty, \infty)$ by

$$h(x) = \begin{cases} 0 & \text{if } x < -1 \quad \text{or} \quad x > 1, \\ \sqrt{1 - x^2} \, x & \text{if } -1 \leq x \leq 1. \end{cases}$$

Let $u = 1 - x^2$. Then $du = -2x \, dx$, $dx = -du/2x$. At $x = 0$, $u = 1$. At $x = 1$, $u = 0$. Therefore

$$\int_0^1 \sqrt{1 - x^2} \, x \, dx = \int_1^0 \sqrt{u} \cdot (-\tfrac{1}{2} \, du) = \int_1^0 -\tfrac{1}{2}\sqrt{u} \, du$$

$$= \tfrac{1}{2} \int_0^1 \sqrt{u} \, du = \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} \bigg]_0^1 = \tfrac{1}{3} - 0 = \tfrac{1}{3}.$$

We see in Figure 4.4.4 that as x increases from 0 to 1, u decreases from 1 to 0, so the limits become reversed. The areas shown in Figure 4.4.5 are equal.

Figure 4.4.4

$$\int_0^1 \sqrt{1 - x^2}\, x\, dx$$

$$\int_1^0 -\tfrac{1}{2} \sqrt{u}\, du$$
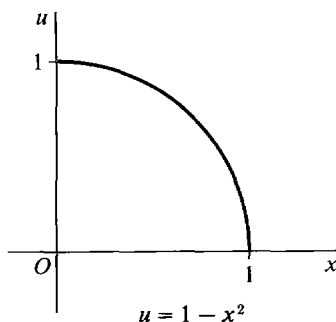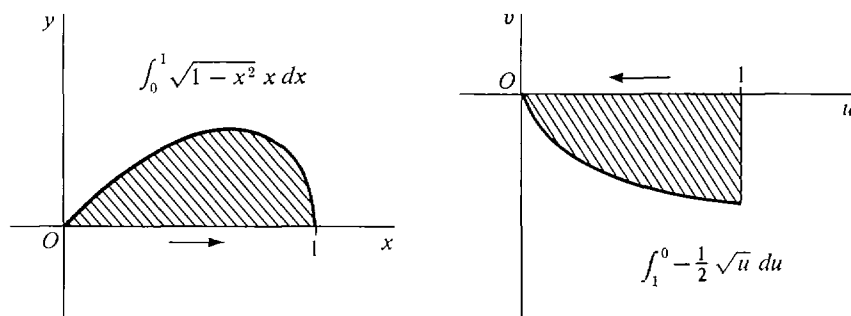
Figure 4.4.5

We can use integration by change of variables to derive the formula for the area of a circle, $A = \pi r^2$, where r is the radius. It is easier to work with a semicircle because the semicircle of radius r is just the region under the curve

$$y = \sqrt{r^2 - x^2}, \qquad -r \le x \le r.$$

To start with we need to give a rigorous definition of π. By definition, π is the area of a unit circle. Thus π is twice the area of the unit semicircle, which means:

DEFINITION

$$\pi = 2 \int_{-1}^{1} \sqrt{1 - x^2}\, dx.$$

The area of a semicircle of radius r is the definite integral

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx.$$

To evaluate this integral we let $x = ru$. Then $dx = r\, du$. When $x = \pm r, u = \pm 1$. Thus

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \int_{-1}^{1} \sqrt{r^2 - (ru)^2}\, r\, du = \int_{-1}^{1} r^2 \sqrt{1 - u^2}\, du$$

$$= r^2 \int_{-1}^{1} \sqrt{1 - u^2}\, du = r^2 \cdot \frac{\pi}{2}.$$

Therefore the semicircle has area $\pi r^2/2$ and the circle area πr^2 (Figure 4.4.6).

EXAMPLE 8   Find $\displaystyle \int_0^1 \frac{3x^2 - 1}{1 + \sqrt{x - x^3}}\, dx.$

Let $u = x - x^3$. Then $du = (1 - 3x^2)\, dx$. When $x = 0$, $u = 0 - 0^3 = 0$. When $x = 1, u = 1 - 1^3 = 0$. Then

$$\int_0^1 \frac{3x^2 - 1}{1 + \sqrt{x - x^3}}\, dx = \int_0^0 -\frac{du}{1 + \sqrt{u}} = 0.$$

As x goes from 0 to 1, u starts at 0, increases for a time, then drops back to 0 (Figure 4.4.7).

$$\frac{\pi r^2}{2} = \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx \qquad \frac{\pi r^2}{2} = \int_{-1}^{1} r^2 \sqrt{1 - u^2} \, du$$

Figure 4.4.6

$u = x - x^3$
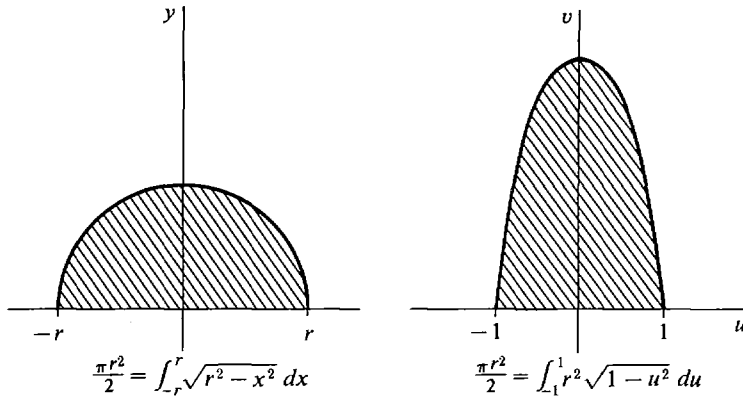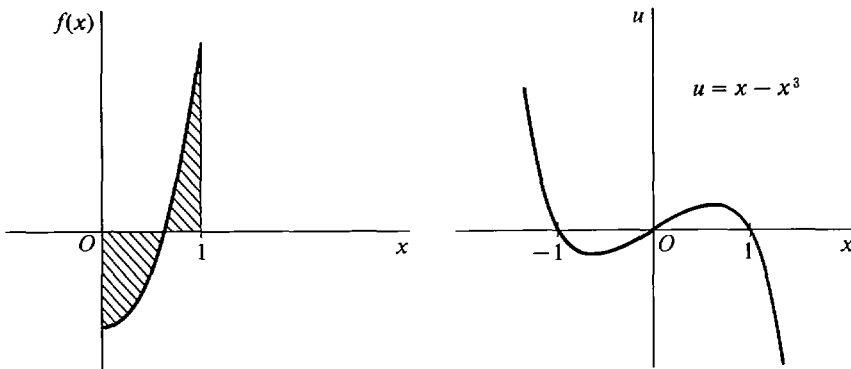
Figure 4.4.7

We do not know how to find the indefinite integrals in this example. Nevertheless the answer is 0 because on changing variables both limits of integration become the same. Using the Addition Property, we can also see that, for instance,

$$\int_{0}^{2/3} \frac{3x^2 - 1}{1 + \sqrt{x - x^3}} \, dx = -\int_{2/3}^{1} \frac{3x^2 - 1}{1 + \sqrt{x - x^3}} \, dx.$$

PROBLEMS FOR SECTION 4.4

In Problems 1–90, evaluate the integral.

1 $\displaystyle \int \frac{1}{(2x + 1)^2} \, dx$

2 $\displaystyle \int \sqrt{3y + 1} \, dy$

3 $\displaystyle \int (3 - 4z)^6 \, dz$

4 $\displaystyle \int (1 - x)^{3/2} \, dx$

5 $\displaystyle \int 2t\sqrt{1 - t^2} \, dt$

6 $\displaystyle \int \frac{x}{\sqrt{2x^2 + 1}} \, dx$

7 $\displaystyle \int x(4 + 5x^2)^2 \, dx$

8 $\displaystyle \int \frac{4y}{(2 + 3y^2)^2} \, dy$

9 $\displaystyle\int \sin(3x)\,dx$

10 $\displaystyle\int \cos(4-2x)\,dx$

11 $\displaystyle\int 6\sin(4x-1)\,dx$

12 $\displaystyle\int a\sin x + b\cos x\,dx$

13 $\displaystyle\int \sin\theta\cos\theta\,d\theta$

14 $\displaystyle\int \sin^2\theta\cos\theta\,d\theta$

15 $\displaystyle\int \cos^3\theta\sin\theta\,d\theta$

16 $\displaystyle\int \sin(2\theta)+\cos(3\theta)\,d\theta$

17 $\displaystyle\int x\sin(x^2+1)\,dx$

18 $\displaystyle\int x^2\cos(x^3)\,dx$

19 $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

20 $\displaystyle\int e^t\cos(e^t)\,dt$

21 $\displaystyle\int \sqrt{\sin t}\cos t\,dt$

22 $\displaystyle\int \sqrt{t}\cos(t\sqrt{t})\,dt$

23 $\displaystyle\int e^{2x}\,dx$

24 $\displaystyle\int 3e^{1-x}\,dx$

25 $\displaystyle\int ae^x + be^{-x}\,dx$

26 $\displaystyle\int (e^x+1)^2\,dx$

27 $\displaystyle\int xe^{x^2}\,dx$

28 $\displaystyle\int xe^{1-x^2}\,dx$

29 $\displaystyle\int be^{ax}\,dx$

30 $\displaystyle\int e^{ax+b}\,dx$

31 $\displaystyle\int e^{\sin\theta}\cos\theta\,d\theta$

32 $\displaystyle\int e^t\sqrt{1+e^t}\,dt$

33 $\displaystyle\int \frac{1}{x+2}\,dx$

34 $\displaystyle\int \frac{2}{3-4x}\,dx$

35 $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

36 $\displaystyle\int \frac{x}{x^2+1}\,dx$

37 $\displaystyle\int \frac{x}{x+1}\,dx$

38 $\displaystyle\int \frac{1}{x^2(1+1/x)}\,dx$

39 $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$

40 $\displaystyle\int \frac{1-2t}{1+2t}\,dt$

41 $\displaystyle\int \frac{3t+4}{5-t}\,dt$

42 $\displaystyle\int \frac{x^2}{x^3+2}\,dx$

43 $\displaystyle\int x^3\sqrt{x^4+5}\,dx$

44 $\displaystyle\int \frac{1}{(2x-1)\sqrt{1-2x}}\,dx$

45 $\displaystyle\int y\sqrt{2+y^2}\,dy$

46 $\displaystyle\int \frac{t}{(2t^2+1)^3}\,dt$

47 $\displaystyle\int \frac{u}{\sqrt{1-u^2}}\,du$

48 $\displaystyle\int (4x+1)\sqrt{2x^2+x+5}\,dx$

49 $\displaystyle\int \frac{1}{\sqrt{3s+2}}\,ds$

50 $\displaystyle\int \sqrt{1-5z}\,dz$

51 $\displaystyle\int \frac{1}{x^2\sqrt{1+1/x}}\,dx$

52 $\displaystyle\int \frac{1}{y^2(1-4/y)^2}\,dy$

53 $\displaystyle\int x^{-3}\sqrt{3+5x^{-2}}\,dx$

54 $\displaystyle\int \frac{1}{\sqrt{x}(1+2\sqrt{x})^2}\,dx$

55 $\displaystyle\int \frac{(3 - \sqrt{x})^2}{\sqrt{x}}\, dx$

56 $\displaystyle\int \frac{\sqrt{1 + \sqrt{t}}}{\sqrt{t}}\, dt$

57 $\displaystyle\int \frac{2 + 1/z}{z^2}\, dz$

58 $\displaystyle\int \frac{1}{(3y + 1)^3}\, dy$

59 $\displaystyle\int \frac{x^2}{\sqrt{x^3 + 4}}\, dx$

60 $\displaystyle\int \frac{x^2}{\sqrt{4x^3 + 1}}\, dx$

61 $\displaystyle\int \frac{x^3}{\sqrt{1 + x^4}}\, dx$

62 $\displaystyle\int x^3\sqrt{2 - x^4}\, dx$

63 $\displaystyle\int t\sqrt{t + 1}\, dt$

64 $\displaystyle\int \frac{s}{(s + 2)^3}\, ds$

65 $\displaystyle\int (2s + 6)(1 - s)^{-4}\, ds$

66 $\displaystyle\int y^3\sqrt{4 + y^2}\, dy$

67 $\displaystyle\int \frac{y^3}{(y^2 + 1)^3}\, dy$

68 $\displaystyle\int \frac{x^5}{\sqrt{1 + x^2}}\, dx$

69 $\displaystyle\int \frac{x}{\sqrt{4x + 1}}\, dx$

70 $\displaystyle\int \sqrt{2 + \sqrt{u}}\, du$

71 $\displaystyle\int u\sqrt{1 - 3u}\, du$

72 $\displaystyle\int \frac{1}{(2\sqrt{x} + 3)^3}\, dx$

73 $\displaystyle\int \frac{4x - 1}{\sqrt{4x + 1}}\, dx$

74 $\displaystyle\int \frac{x^2}{\sqrt{x - 1}}\, dx$

75 $\displaystyle\int \frac{x^3}{1 - x^4}\, dx$

76 $\displaystyle\int \frac{y^3}{2 - y^2}\, dy$

77 $\displaystyle\int \frac{y^5}{1 + y^2}\, dy$

78 $\displaystyle\int \frac{u}{(u + 4)^2}\, du$

79 $\displaystyle\int \frac{6u - 5}{(3u + 2)^2}\, du$

80 $\displaystyle\int \frac{1}{1 + \sqrt{x}}\, dx$

81 $\displaystyle\int \frac{\sqrt{x}}{2 + \sqrt{x}}\, dx$

82 $\displaystyle\int \frac{e^x + \cos x}{e^x + \sin x}\, dx$

83 $\displaystyle\int \frac{\cos\theta}{\sin\theta}\, d\theta$

84 $\displaystyle\int \tan\theta\, d\theta$

85 $\displaystyle\int \frac{1}{a + bx}\, dx$

86 $\displaystyle\int \frac{2x + 1}{x^2 + x + 1}\, dx$

87 $\displaystyle\int \frac{\sin\theta}{1 + \cos\theta}\, d\theta$

88 $\displaystyle\int \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}\, d\theta$

89 $\displaystyle\int \frac{\ln x}{x}\, dx$

90 $\displaystyle\int \frac{1}{x \ln x}\, dx$

In Problems 91–108, evaluate the definite integral.

91 $\displaystyle\int_0^{\pi/3} \sin\theta\, d\theta$

92 $\displaystyle\int_{-\pi/4}^{\pi/4} \cos(2\theta)\, d\theta$

93 $\displaystyle\int_{-1}^{1} e^x\, dx$

94 $\displaystyle\int_0^1 xe^{x^2}\, dx$

95 $\displaystyle\int_1^4 \frac{1}{2x}\, dx$

96 $\displaystyle\int_0^1 \frac{x}{x^2 + 1}\, dx$

97 $\displaystyle\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$

98 $\displaystyle\int_0^{2\pi} a\sin\theta + b\cos\theta\, d\theta$

99 $\displaystyle\int_0^2 \sqrt{x+1}\,dx$

100 $\displaystyle\int_0^1 \frac{1}{(4x+3)^2}\,dx$

101 $\displaystyle\int_0^4 (2x+1)^{3/2}\,dx$

102 $\displaystyle\int_0^1 t(t^2+3)^{-2}\,dt$

103 $\displaystyle\int_0^1 (1+6x)^3\,dx$

104 $\displaystyle\int_1^5 \frac{2}{\sqrt{3t+1}}\,dt$

105 $\displaystyle\int_0^2 v\sqrt{2v^2+9}\,dv$

106 $\displaystyle\int_{-1}^1 \frac{x^2}{(4-x^3)^2}\,dx$

107 $\displaystyle\int_{-1}^1 \frac{x}{2-x^2}\,dx$

108 $\displaystyle\int_{\sqrt6}^5 x(x^2+2)^{1/3}\,dx$

109 Find the area of the region below the curve $y = 1/(10-3x)$ from $x = 1$ to $x = 2$.

110 Find the area of the region under one arch of the curve $y = \sin x \cos x$.

111 Find the area of the region under one arch of the curve $y = \cos(3x)$.

112 Find the area of the region below the curve $y = 4x\sqrt{4-x^2}$ between $x = 0$ and $x = 2$.

113 Find the area below the curve $y = (1+7x)^{2/3}$ between $x = 0$ and $x = 1$.

114 Find the area below the curve $y = x/(x^2+1)^2$ between $x = 0$ and $x = 3$.

☐ **115** Evaluate: $\displaystyle\int_0^1 \frac{1-2x}{1+\sqrt[3]{x-x^2}}\,dx$

☐ **116** Evaluate: $\displaystyle\int_{-1}^1 2x\sqrt{(1-x^2)^3+1}\,dx$

☐ **117** Let f and g have continuous derivatives and evaluate $\int f'(g(x))g'(x)\,dx$.

☐ **118** A real function f is said to be *even* if $f(x) = f(-x)$ for all x. Show that if f is a continuous even function, then $\int_{-b}^0 f(x)\,dx = \int_0^b f(x)\,dx$.

☐ **119** An *odd* function is a real function g such that $g(-x) = -g(x)$ for all x. Prove that for a continuous odd function g, $\int_{-b}^b g(x)\,dx = 0$.

4.5 AREA BETWEEN TWO CURVES

A region in the plane can often be represented as the region between two curves. For example, the unit circle is the region between the curves

$$y = -\sqrt{1-x^2}, \qquad y = \sqrt{1-x^2}, \qquad -1 \le x \le 1$$

shown in Figure 4.5.1. Consider two continuous functions f and g on $[a, b]$ such that $f(x) \le g(x)$ for all x in $[a, b]$. The region R, bounded by the curves

$$y = f(x), \qquad y = g(x), \qquad x = a, \qquad x = b,$$

is called the region between $f(x)$ and $g(x)$ from a to b. If both curves are above the x-axis as in Figure 4.5.2, the area of the region R can be found by subtracting the area below f from the area below g:

$$\text{area of } R = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx.$$

It is usually easier to work with a single integral and write

$$\text{area of } R = \int_a^b (g(x) - f(x))\,dx.$$

Figure 4.5.1   Figure 4.5.2
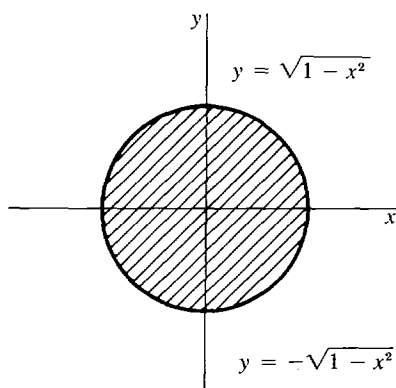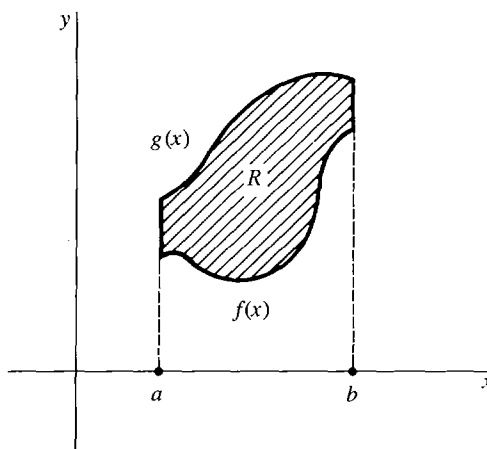
In the general case shown in Figure 4.5.3, we may move the region R above the x-axis by adding a constant c to both $f(x)$ and $g(x)$ without changing the area, and the same formula holds:

$$\text{area of } R = \int_a^b (g(x) + c)\, dx - \int_a^b (f(x) + c)\, dx$$
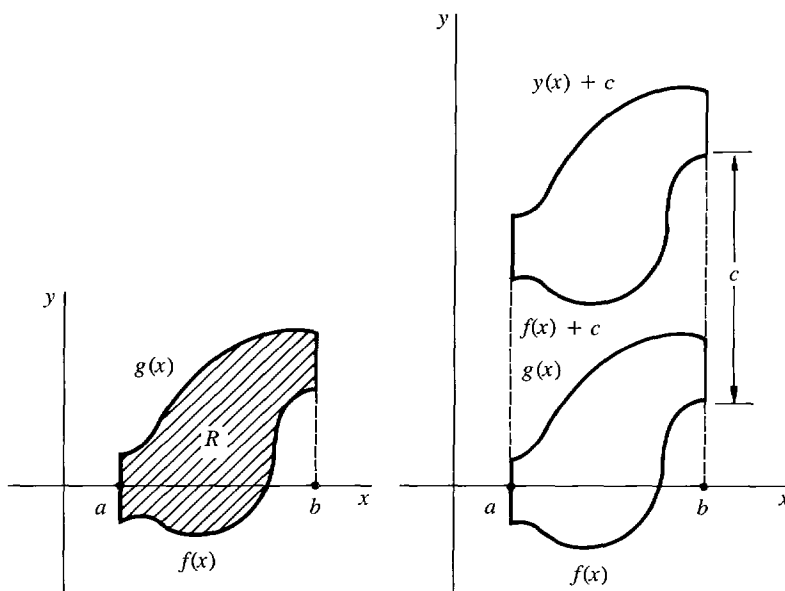
$$= \int_a^b (g(x) - f(x))\, dx.$$

Figure 4.5.3

To sum up, we define the area between two curves as follows.

DEFINITION

If f and g are continuous and $f(x) \leq g(x)$ for $a \leq x \leq b$, then the **area** of the region R between $f(x)$ and $g(x)$ from a to b is defined as

$$\int_a^b (g(x) - f(x))\, dx.$$

EXAMPLE 1   Find the area of the region between the curves $y = \frac{1}{2}x^2 - 1$ and $y = x$ from $x = 1$ to $x = 2$. In Figure 4.5.4, we sketch the curves to check that $\frac{1}{2}x^2 - 1 \leq x$ for $1 \leq x \leq 2$. Then

$$A = \int_1^2 x - (\tfrac{1}{2}x^2 - 1)\, dx = \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + x \Big]_1^2 = \tfrac{8}{6}.$$
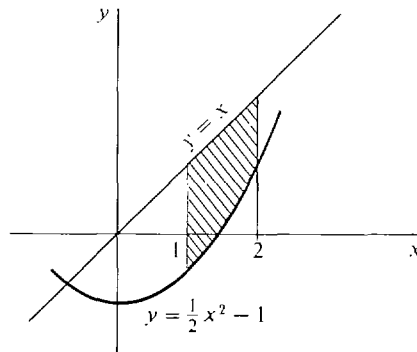
Figure 4.5.4

EXAMPLE 2   Find the area of the region bounded above by $y = x + 2$ and below by $y = x^2$.

Part of the problem is to find the limits of integration. First draw a sketch (Figure 4.5.5). The curves intersect at two points, which can be found by solving the equation $x + 2 = x^2$ for x.

$$x^2 - (x + 2) = 0, \qquad (x + 1)(x - 2) = 0,$$
$$x = -1 \quad \text{and} \quad x = 2.$$

Then $A = \int_{-1}^2 (x + 2 - x^2)\, dx = \tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3 \Big]_{-1}^2 = 4\tfrac{1}{2}.$
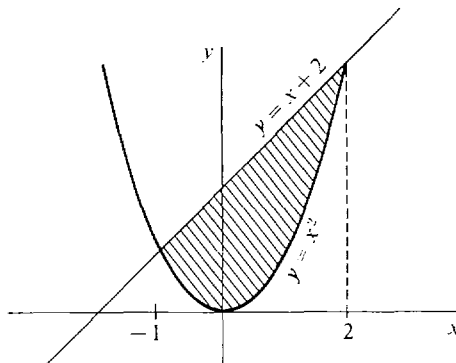
Figure 4.5.5

EXAMPLE 3 Find the area of the region R bounded below by the line $y = -1$ and above by the curves $y = x^3$ and $y = 2 - x$. The region is shown in Figure 4.5.6.
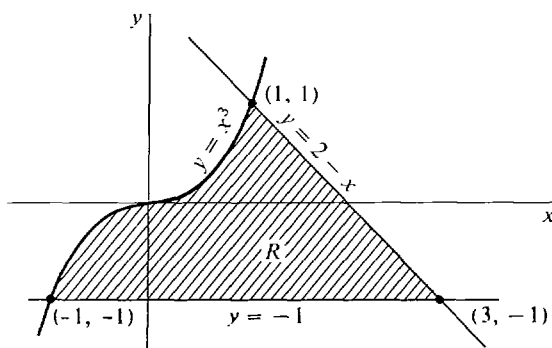
Figure 4.5.6

This problem can be solved in three ways. Each solution illustrates a different trick which is useful in other area problems. The three corners of the region are:

$(-1, -1)$,	where $y = x^3$ and $y = -1$ cross.
$(3, -1)$,	where $y = 2 - x$ and $y = -1$ cross.
$(1, 1)$,	where $y = x^3$ and $y = 2 - x$ cross.

Note that $y = x^3$ and $y = 2 - x$ can cross at only one point because x^3 is always increasing and $2 - x$ is always decreasing.

FIRST SOLUTION Break the region into the two parts shown in Figure 4.5.7: R_1 from $x = -1$ to $x = 1$, and R_2 from $x = 1$ to $x = 3$. Then

$$\text{area of } R = \text{area of } R_1 + \text{area of } R_2.$$

$$\text{area of } R_1 = \int_{-1}^{1} x^3 - (-1)\, dx = \tfrac{1}{4}x^4 + x \Big]_{-1}^{1} = 2.$$

$$\text{area of } R_2 = \int_{1}^{3} (2 - x) - (-1)\, dx = 3x - \tfrac{1}{2}x^2 \Big]_{1}^{3} = 2.$$

$$\text{area of } R = 2 + 2 = 4.$$
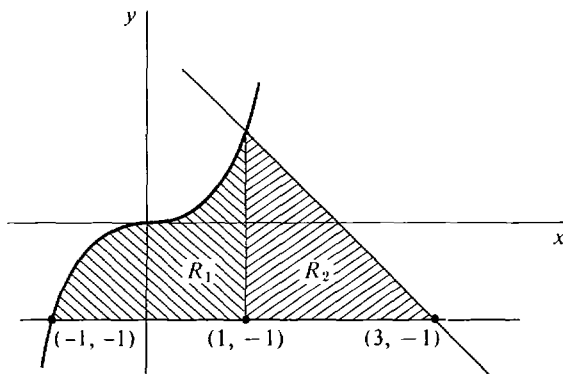
Figure 4.5.7   First solution

SECOND SOLUTION   Form the triangular region S between $y = -1$ and $y = 2 - x$ from -1 to 3. The region R is obtained by subtracting from S the region S_1 shown in Figure 4.5.8. Then

$$\text{area of } R = \text{area of } S - \text{area of } S_1.$$

$$\text{area of } S = \int_{-1}^{3} (2 - x) - (-1)\, dx = 3x - \tfrac{1}{2}x^2 \Big]_{-1}^{3} = 8.$$

$$\text{area of } S_1 = \int_{-1}^{1} (2 - x) - x^3\, dx = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 \Big]_{-1}^{1} = 4.$$

$$\text{area of } R = 8 - 4 = 4.$$
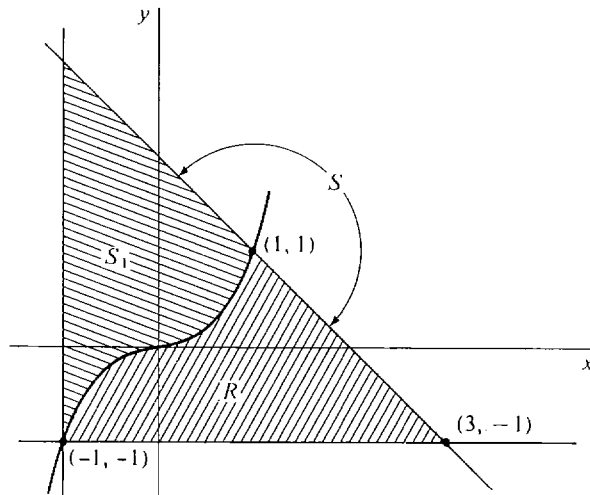
Figure 4.5.8   Second solution

THIRD SOLUTION   Use y as the independent variable and x as the dependent variable. Write the boundary curves with x as a function of y.

$$y = 2 - x \qquad \text{becomes } x = 2 - y.$$
$$y = x^3 \qquad \text{becomes } x = y^{1/3}.$$

The limits of integration are $y = -1$ and $y = 1$ (see Figure 4.5.9). Then

$$A = \int_{-1}^{1} (2 - y) - y^{1/3}\, dy = 2y - \tfrac{1}{2}y^2 - \tfrac{3}{4}y^{4/3} \Big]_{-1}^{1} = 4.$$

As expected, all three solutions gave the same answer.

Figure 4.5.9   Third solution

PROBLEMS FOR SECTION 4.5

In Problems 1–43 below, sketch the given curves and find the area of the region bounded by them.

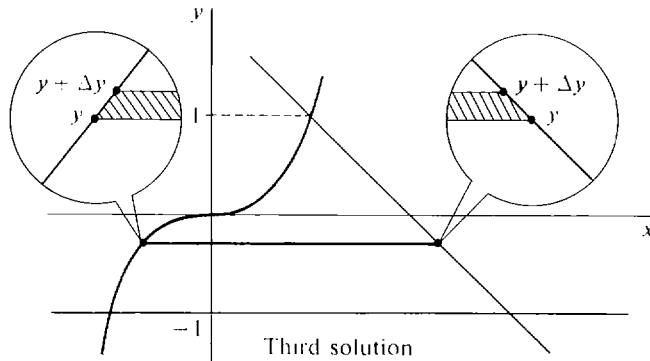|	
1	$f(x) = 0, \quad g(x) = 5x - x^2, \quad 0 \le x \le 4$
2	$f(x) = \sqrt{x}, \quad g(x) = x^2, \quad 1 \le x \le 4$
3	$f(x) = x\sqrt{1 - x^2}, \quad g(x) = 1, \quad -1 \le x \le 1$
4	$y = x - 2, \quad y = 3x^{1/3}, \quad 0 \le x \le 1$
5	$y = \sqrt{x}, \quad y = \sqrt{x + 1}, \quad 0 \le x \le 4$
6	$y = \sqrt{x^2 + 1} - x, \quad y = \sqrt{x^2 + 1} + x, \quad -0 \le x \le 1$
7	The x-axis and the curve $y = -5 + 6x - x^2$
8	The x-axis and the curve $y = 1 - x^4$
9	The y-axis and the curve $x = 25 - y^2$
10	The y-axis and the curve $x = y(8 - y)$
11	$y = \cos x, \quad y = 2 \cos x, \quad -\pi/2 \le x \le \pi/2$
12	$y = \sin x \cos x, \quad y = 1, \quad 0 \le x \le \pi/2$
13	$y = -\sin x, \quad y = \sin x, \quad 0 \le x \le \pi$
14	$y = \sin x, \quad y = \cos x, \quad 0 \le x \le \pi/4$
15	$y = \sin x \cos x, \quad y = \sin x, \quad 0 \le x \le \pi$
16	$y = \sin^2 x \cos x, \quad y = \sin x \cos x, \quad 0 \le x \le \pi/2$
17	$y = x, \quad y = e^x, \quad 0 \le x \le 2$
18	$y = e^{-x}, \quad y = e^x, \quad 0 \le x \le 2$
19	$y = -e^x, \quad y = e^x, \quad -1 \le x \le 1$
20	$y = xe^{x^2}, \quad y = e, \quad 0 \le x \le 1$
21	$y = \dfrac{1}{x + 1}, \quad y = 1, \quad 0 \le x \le 2$
22	$y = \dfrac{1}{2x + 1}, \quad y = \dfrac{1}{x + 1}, \quad 0 \le x \le 2$
23	$y = 1/x, \quad y = x, \quad 1 \le x \le 2$
24	$y = \dfrac{x}{x^2 + 1}, \quad y = \tfrac{1}{2}, \quad 0 \le x \le 1$
25	$f(x) = x^{3/2}, \quad g(x) = x^{2/3}$
26	$y = x^2 - 2x, \quad y = x - 2$
27	$y = x^4 - 2x^2, \quad y = 2x^2 + 12$
28	$y = x^4 - 1, \quad y = x^3 - x$
29	$y = x^4/(x^2 + 1), \quad y = 1/(x^2 + 1)$
30	$y = x^3\sqrt{1 - x^2}, \quad y = x\sqrt{1 - x^2}, \quad 0 \le x$
31	$y = 2x^2, \quad y = x^2 + 4$
32	$x = y^2, \quad x = 2 - y^2$
33	$\sqrt{x} + \sqrt{y} = 1$ and the x- and y-axes
34	$x^2 y = 4, \quad x^2 + y = 5 \quad$ (first quadrant)
35	$y = x\sqrt{x + 1}, \quad y = 2x$
36	$y = 0, \quad y = x^3 + x + 2, \quad x = 2$
37	$y = 2x + 4, \quad y = 2 - 3x, \quad y = -x$
38	$y = x^2 - 1, \quad y = (x - 1)^2, \quad y = (x + 1)^2$

39 $y = \sqrt{x},\quad y = 1,\quad y = 10 - 2x$

40 $y = x - 2,\quad y = 2 - x,\quad y = \sqrt{x}$

41 $y = -x,\quad y = \sqrt[3]{x},\quad y = 3x - 2$

42 $y = -2,\quad y = x^3 + x,\quad x + y = 3$

43 $y = x^2,\quad y = 2x^{-2},\quad y = 2x^{-3}$ (first quadrant)

44   Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Use the fact that the unit circle has area π.

45   Sketch the four-sided region bounded by the lines $y = 1$, $y = x$, $y = 2x$, and $y = 6 - x$ and find its area.

46   Find the number $c > 0$ such that the region bounded by the curves $y = x$, $y = -2x$, and $x = c$ has area 6.

47   Find the number $c > 1$ such that the region bounded by the curves $y = 1$, $y = x^{-2}$, and $x = c$ has area 1.

48   Find the number c such that the region bounded by the curves $y = x^2$ and $y = c$ has area 36.

49   Find the number $c > 0$ such that the region bounded by the curves $y = x^2$ and $y = cx$ has area 9.

50   Find the value of c between -1 and 2 such that the area of the region bounded by the lines $y = -x$, $y = 2x$, and $y = 1 + cx$ is a minimum.

51   Find the value of c such that the line $y = c$ bisects the region bounded by the curves $y = x^2$ and $y = 1$.

52   Find the value of c such that the line $y = cx$ bisects the region bounded by the x-axis and the curve $y = x - x^2$.

4.6 NUMERICAL INTEGRATION

In numerical integration, one computes an approximate value for the definite integral rather than finding an exact value. In this section we shall present two methods of numerical integration, called the Trapezoidal Rule and Simpson's Rule.

The Fundamental Theorem of Calculus gives us a method of computing the definite integral of a given continuous function f from a to b. The method is to find, by trial and error, an antiderivative F of f and then to use the equation

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

When the method works, it provides an exact value for the integral. However, the method succeeds only if the antiderivative happens to be a function that can be described in a simple way. For many integrals one cannot find a formula for the antiderivative, and the method fails. Such integrals can still be computed approximately using numerical integration.

The Trapezoidal Rule and Simpson's Rule can always be applied and do not use the antiderivative. They are easy to carry out on a computer or hand calculator. We already discussed one method of approximating the definite integral in Section 4.1, the Riemann sum. The Trapezoidal Rule is a modified form of the Riemann sum, which gives a much closer approximation for a given amount of effort. Simpson's Rule is a further modification that gives still better approximations.

Let f be a continuous function on an interval I, and let $a < b$ in I. By definition, for each positive infinitesimal dx the definite integral

$$\int_a^b f(x)\,dx$$

is the standard part of the infinite Riemann sum

$$\sum_a^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx = st\left[\sum_a^b f(x)\,dx\right].$$

In Section 4.1, examples were worked out to show that the finite Riemann sums become very close to the definite integral when Δx is small; that is, the finite Riemann sums approximate the definite integral. In Section 4.2, we saw that the definite integral is the limit of the finite Riemann sums as $\Delta x \to 0^+$:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0^+} \sum_a^b f(x)\,\Delta x.$$

The Riemann sum, which is a sum of areas of rectangles, is a rather inefficient approximation of the definite integral. We can usually get a much closer approximation with the same amount of work by adding up areas of trapezoids instead of rectangles, forming the Trapezoidal Rule suggested by Figure 4.6.1. The Trapezoidal Rule also provides a formula, called an error estimate, which tells us how close the approximation is to the exact value of the definite integral.

Riemann Sum   Trapezoidal Approximation
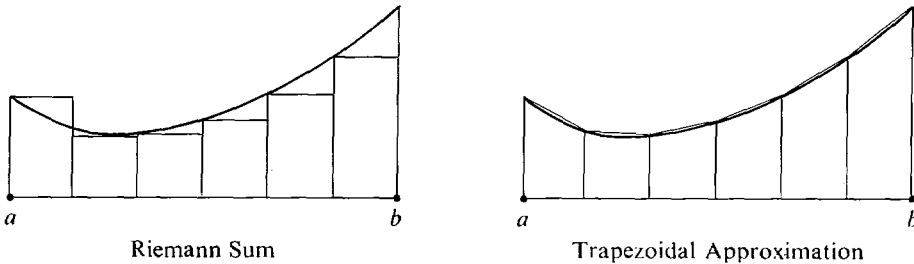
Figure 4.6.1

Choose a positive integer n and divide the interval $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. The partition points are $a = x_0, x_1, x_2, \ldots, x_n = b$.

The trapezoidal approximation is the area of the region under the broken line connecting the points

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

Since all of these points lie on the curve $y = f(x)$, the broken line closely follows the curve. So one would expect the area of the region under the broken line to closely approximate the area under the curve.

Consider a single subinterval $[x_m, x_{m+1}]$ of width Δx. The region under the line segment connecting the two points

$$(x_m, f(x_m)), \qquad (x_{m+1}, f(x_{m+1}))$$

is a trapezoid and its area is

$$\frac{f(x_m) + f(x_{m+1})}{2} \Delta x.$$

The sum of the areas of the trapezoids is a modified Riemann sum

$$\sum_a^b \frac{f(x) + f(x + \Delta x)}{2} \Delta x$$

$$= \left(\frac{f(x_0) + f(x_1)}{2} + \frac{f(x_1) + f(x_2)}{2} + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2} \right) \Delta x$$

$$= [\tfrac{1}{2} f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2} f(x_n)] \Delta x.$$

We thus make the definition:

DEFINITION

Let $\Delta x = (b - a)/n$ evenly divide $b - a$. Then by the **trapezoidal approximation** to the definite integral $\int_a^b f(x) \, dx$ we mean the sum

$$\sum_a^b \frac{f(x) + f(x + \Delta x)}{2} \Delta x = [\tfrac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \tfrac{1}{2} f(x_n)] \Delta x.$$

The Trapezoidal Approximation of an integral $\int_a^b f(x) \, dx$ can be computed very efficiently on most hand calculators. First compute the sum

$$\tfrac{1}{2} f(x_0) + f(x_1) + f(x_2) + \cdots + \tfrac{1}{2} f(x_n)$$

by cumulative addition. Then multiply this sum by Δx to obtain the Trapezoidal Approximation.

THEOREM 1

For a continuous function f on $[a, b]$, the trapezoidal approximation approaches the definite integral as $\Delta x \to 0^+$, that is,

$$\int_a^b f(x) \, dx = \lim_{\Delta x \to 0^+} \sum_a^b \frac{f(x) + f(x + \Delta x)}{2} \Delta x.$$

PROOF Comparing the formulas for the trapezoidal approximation and the Riemann sum, we see that

$$\sum_a^b \frac{f(x) + f(x + \Delta x)}{2} \Delta x = \sum_a^b f(x) \Delta x + (\tfrac{1}{2} f(x_n) - \tfrac{1}{2} f(x_0)) \Delta x.$$

For dx positive infinitesimal, the extra term

$$(\tfrac{1}{2} f(x_H) - \tfrac{1}{2} f(x_0)) \, dx$$

is infinitely small. It follows that

$$\sum_a^b \frac{f(x) + f(x + dx)}{2} \, dx \approx \sum_a^b f(x) \, dx \approx \int_a^b f(x) \, dx.$$

From a practical standpoint, it is desirable to have a good estimate of error. We shall first work an example and then state a theorem which gives an error estimate for the trapezoidal approximation.

EXAMPLE 1 Approximate the definite integral

$$\int_0^1 \sqrt{1 + x^2}\, dx.$$

Use the trapezoidal approximation with $\Delta x = \frac{1}{5}$. We first make a table of values of $\sqrt{1 + x^2}$. The graph is drawn in Figure 4.6.2.
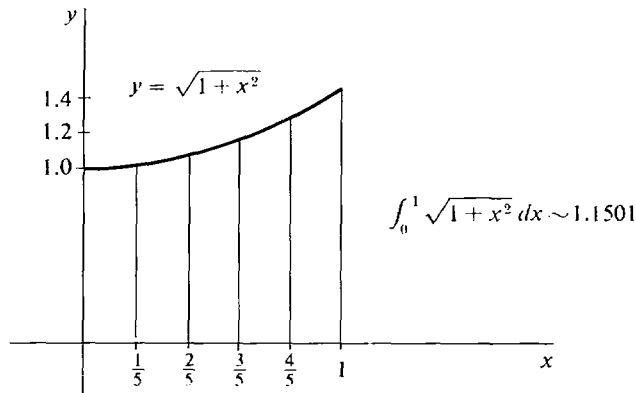

Figure 4.6.2

x	$\sqrt{1 + x^2}$	$\sqrt{1 + x^2}$ to four places	term in trapezoidal approximation
$x_0 = 0$	1	1.0000	$0.5000 = \frac{1}{2}f(x_0)$
$x_1 = \frac{1}{5}$	$\sqrt{1.04}$	1.0198	$1.0198 = f(x_1)$
$x_2 = \frac{2}{5}$	$\sqrt{1.16}$	1.0770	$1.0770 = f(x_2)$
$x_3 = \frac{3}{5}$	$\sqrt{1.36}$	1.1662	$1.1662 = f(x_3)$
$x_4 = \frac{4}{5}$	$\sqrt{1.64}$	1.2806	$1.2806 = f(x_4)$
$x_5 = 1$	$\sqrt{2}$	1.4142	$0.7071 = \frac{1}{2}f(x_5)$
			$5.7507 = \text{total}$

Thus, $\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + \frac{1}{2}f(x_5) = 5.7507$. Since $\Delta x = \frac{1}{5}$, the trapezoidal approximation is

$$(5.7507) \cdot \tfrac{1}{5} = 1.1501,$$

$$\int_0^1 \sqrt{1 + x^2}\, dx \sim 1.1501.$$

The trapezoidal approximation can be made as close to the definite integral as we want by taking Δx small. From a practical standpoint, however, it is helpful to know how small we should take Δx in order to be sure of a given degree of accuracy. For instance, suppose we need to know the definite integral to three decimal places. How small must we take Δx in our trapezoidal approximation? The answer is given

by the Trapezoidal Rule, which gives an error estimate for the trapezoidal approximation.

The *error* in the trapezoidal approximation is the absolute value of the difference between the trapezoidal sum and the definite integral,

$$\text{error} = \left| \sum_a^b \frac{f(x) + f(x + \Delta x)}{2} \Delta x - \int_a^b f(x)\, dx \right|.$$

An *error estimate* for the trapezoidal approximation is a function $E(\Delta x)$, which is known to be greater than or equal to the error.

Thus if $E(\Delta x)$ is an error estimate, the trapezoidal sum is within $E(\Delta x)$ of the definite integral. If we want to be sure that the trapezoidal approximation is accurate to three decimal places—i.e., the error is less than 0.0005—we choose Δx so that $E(\Delta x) \leq 0.0005$. We are now ready to state the Trapezoidal Rule.

TRAPEZOIDAL RULE

Let f be a function whose second derivative f″ exists and has absolute value at most M on a closed interval $[a, b]$,

$$|f''(x)| \leq M \quad \text{for } a \leq x \leq b.$$

If Δx evenly divides $b - a$, then the trapezoidal approximation of the definite integral of f has the error estimate

$$\frac{b - a}{12} M(\Delta x)^2.$$

That is,

$$\left| \sum_a^b \frac{f(x) + f(x + \Delta x)}{2} \Delta x - \int_a^b f(x)\, dx \right| \leq \frac{b - a}{12} M(\Delta x)^2.$$

The proof is omitted.

EXAMPLE 1 (Concluded)   We let $f(x) = \sqrt{1 + x^2}$.   Then

$$f'(x) = \frac{x}{\sqrt{1 + x^2}},$$
$$f''(x) = \frac{\sqrt{1 + x^2} - x^2/\sqrt{1 + x^2}}{1 + x^2} = \frac{1}{(1 + x^2)^{3/2}}.$$

Therefore $|f''(x)| \leq 1$ for all x in $[0, 1]$. We take $M = 1$ and use the error estimate given by the Trapezoidal Rule,

$$\frac{b - a}{12} M(\Delta x)^2 = \frac{1}{12} \cdot 1 \cdot \left(\frac{1}{5}\right)^2 = \frac{1}{300}.$$

Thus our approximation is within an accuracy of $1/300$,

$$\left| \int_0^1 \sqrt{1 + x^2}\, dx - 1.150 \right| \leq 1/300 \sim 0.0033.$$

This shows that the integral is, at least, between 1.146 and 1.154.

In this particular example we can even conclude that the integral is between 1.146 and 1.150 (rounded off to three places). That is, the integral is less than its trapezoidal approximation. This is because the second derivative $f''(x) = (1 + x^2)^{-3/2}$ is always greater than 0, whence the curve is concave upwards and therefore $y = f(x)$ is always less than or equal to the broken line used in the trapezoidal approximation. Actually, the value to three places is 1.148. This can be found by taking $\Delta x = \frac{1}{10}$.

EXAMPLE 2 Consider the integral

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \pi/2.$$

Let
$$f(x) = \sqrt{1 - x^2}.$$

By Theorem 1, we have

$$\lim_{\Delta x \to 0^+} \sum_{-1}^{1} \frac{f(x) + f(x + \Delta x)}{2} \Delta x = \frac{\pi}{2}.$$

However, the Trapezoidal Rule fails to give an error estimate in this case because $f'(x)$ is discontinuous at $x = \pm 1$.

We now turn to Simpson's Rule, for which the number of subintervals n must be even. As before, we divide the interval $[a, b]$ into n subintervals of equal length Δx with the $n + 1$ partition points

$$a = x_0, x_1, \ldots, x_n = b.$$

We shall use subintervals of length $2 \Delta x$ rather than Δx. On each of the $n/2$ subintervals

$$[x_0, x_2], [x_2, x_4], \ldots, [x_{n-2}, x_n],$$

of length $2 \Delta x$ we approximate the curve $y = f(x)$ by a parabolic arc that meets the curve at both endpoints and the midpoint of the subinterval, as shown in Figure 4.6.3. We then add up the areas under each of the parabolic arcs to obtain an approximation to the area under the curve, which is the definite integral. We begin with a lemma that gives a formula for the area of the region under one parabolic arc.
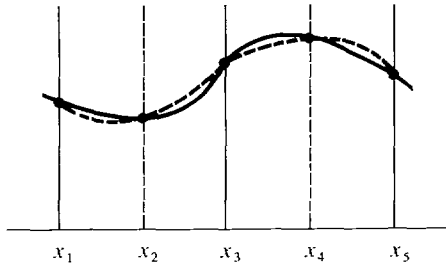

Figure 4.6.3

LEMMA

The area of the region under the parabola through three points (u, r), $(u + h, s)$, and $(u + 2h, t)$ (shown in Figure 4.6.4) is
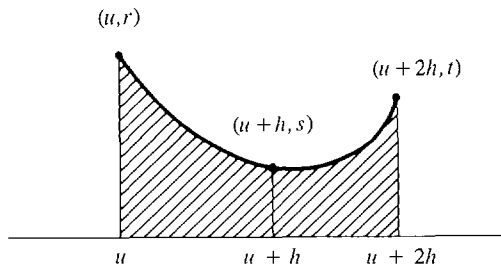
Figure 4.6.4

$$\frac{h}{3}(r + 4s + t).$$

The lemma is proved at the end of this section. Using the lemma, we find that the area of the region under one parabolic arc from x_k to x_{k+2} is

$$\frac{\Delta x}{3}[f(x_k) + 4f(x_{k+1}) + f(x_{k+2})].$$

It follows that the sum of the $n/2$ regions under the parabolic arcs is a modified Riemann sum,

$$\sum_a^b \left[\frac{f(x) + 4f(x + \Delta x) + f(x + 2\,\Delta x)}{3} \right] \Delta x$$

$$= \frac{\Delta x}{3} \{[f(x_0) + 4f(x_1) + f(x_2)] + [f(x_2) + 4f(x_3) + f(x_4)] + \cdots \}$$

$$= \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

This modified Riemann sum is *Simpson's approximation* to the definite integral. Note the sequence of coefficients,

$$1, 4, 2, 4, 2, \ldots, 2, 4, 1.$$

Like the trapezoidal approximation, it is easily computed on a computer or hand calculator.

THEOREM 2

For a continuous function f on $[a, b]$, Simpson's approximation approaches the definite integral as $\Delta x \to 0^+$,

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0^+} \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + f(x_n)].$$

Simpson's approximation is almost as easy to calculate as the trapezoidal approximation, but is much more accurate. Simpson's Rule is an error estimate that involves the fourth derivative of the function and the fourth power of Δx.

SIMPSON'S RULE

Suppose the function f has a fourth derivative on the interval $[a, b]$ that has absolute value at most M,

$$|f^{(4)}(x)| \leq M \text{ for } a \leq x \leq b.$$

If $[a, b]$ is divided into an even number of subintervals of length Δx, then Simpson's approximation to the definite integral has the error estimate

$$\frac{b - a}{180} M(\Delta x)^4.$$

EXAMPLE 3   Use Simpson's Rule with $\Delta x = 0.25$ to approximate the integral

$$A = \int_0^1 e^{-x^2/2} \, dx$$

and find the error estimate.

The curve is the normal (bell-shaped) curve used in statistics, shown in Figure 4.6.5. We are to divide the interval $[0, 1]$ into four subintervals of equal length $\Delta x = 0.25$. The following table shows the values of x and y and the coefficient to be used in Simpson's approximation for each partition point.
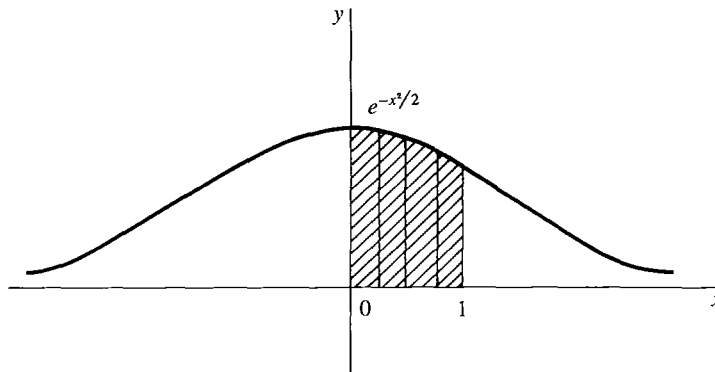
Figure 4.6.5   Example 3

x	$e^{-x^2/2}$	Coefficient
0.0	1.000000	1
0.25	0.969233	4
0.5	0.882496	2
0.75	0.754840	4
1.0	0.606531	1

The sum used in the Simpson approximation is then

$[1.000000 + 4 \cdot (0.969233) + 2 \cdot (0.882496) + 4 \cdot (0.754840) + 0.606531]$
  $= 10.267816$

To get the Simpson approximation, we multiply this sum by $\Delta x/3$:

$$S = (10.267816) \cdot (0.25)/3 = 0.855651.$$

To find the error estimate we need the fourth derivative of

$$y = e^{-x^2/2}.$$

The fourth derivative can be computed as usual and turns out to be

$$y^{(4)} = (x^4 - 6x^2 + 3)e^{-x^2/2}.$$

On the interval $[0, 1]$, $y^{(4)}$ is decreasing because both $x^4 - 6x^2 + 3$ and $-x^2/2$ are decreasing, and therefore $y^{(4)}$ has its maximum value at $x = 0$ and its minimum value at $x = 1$,

$$\text{maximum:} \qquad y^{(4)}(0) = 3$$
$$\text{minimum:} \qquad y^{(4)}(1) = -1.213061$$

The maximum value of the absolute value $|y^{(4)}|$ is thus $M = 3$. The error estimate in Simpson's Rule is then

$$\frac{b - a}{180}(\Delta x)^4 M = \frac{1}{180} \cdot (0.25)^4 \cdot 3 = 0.000065.$$

This shows that the integral is within 0.000065 of the approximation; that is,

$$\int_0^1 e^{-x^2/2}\, dx = 0.855651 \pm 0.000065,$$

or using inequalities,

$$0.855586 \leq \int_0^1 e^{-x^2/2}\, dx \leq 0.855716.$$

For comparison, a more accurate computation with a smaller Δx shows that the actual value to six places is

$$\int_0^1 e^{-x^2/2}\, dx = 0.855624.$$

The Trapezoidal Rule for this integral and the same value of $\Delta x = 0.25$ give an approximate value of 0.85246 for the integral and an error estimate of 0.00521.

PROOF OF THE LEMMA   The algebra is simpler if the y-axis is drawn through the second point, so that $u + h = 0$, and the three points have coordinates

$$(-h, r), (0, s), (h, t).$$

Suppose the parabola has the equation $y = ax^2 + bx + c$. Then the area under the parabola is

$$A = \int_{-h}^{h} (ax^2 + bx + c)\, dx$$
$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Big]_{-h}^{h}$$
$$= \frac{2}{3}ah^3 + 2ch.$$

When we substitute the coordinates of the three points $(-h, r)$, $(0, s)$, (h, t) into the equation for the parabola, we obtain the three equations

$$r = ah^2 - bh + c,$$
$$s = c,$$
$$t = ah^2 + bh + c.$$

Add the first and third equations and solve for a:

$$r + t = 2ah^2 + 2c$$
$$a = \frac{r + t - 2c}{2h^2}$$

Finally, substitute the above expression for a and s for c in the equation for the area:

$$A = \frac{2}{3} ah^3 + 2ch$$
$$= \frac{r + t - 2c}{2h^2} \cdot \frac{2}{3} \cdot h^3 + 2ch$$
$$= \frac{r + t - 2c + 6c}{3} \cdot h$$
$$= \frac{h}{3} (r + 4c + t).$$
$$= \frac{h}{3} (r + 4s + t).$$

PROBLEMS FOR SECTION 4.6

Approximate the integrals in Problems 1–20 using (a) the Trapezoidal Rule and (b) Simpson's Rule. When possible, find error estimates. If a hand calculator is available, do the problems again with $\Delta x = 0.1$.

1 $\displaystyle\int_0^2 x \, dx, \quad \Delta x = 0.5$ 2 $\displaystyle\int_0^2 x^3 dx, \quad \Delta x = 0.5$

3 $\displaystyle\int_1^2 \sqrt{x^2 - 1} \, dx, \quad \Delta x = 0.25$ 4 $\displaystyle\int_1^3 \frac{1}{x} dx, \quad \Delta x = 0.5$

5 $\displaystyle\int_1^2 \frac{1}{1 + x^2} dx, \quad \Delta x = 0.25$ 6 $\displaystyle\int_0^1 x\sqrt{x + 1} \, dx, \quad \Delta x = 0.25$

7 $\displaystyle\int_1^5 \frac{x}{x + 1} dx, \quad \Delta x = 0.5$ 8 $\displaystyle\int_0^1 \sqrt{x^3 + 1} \, dx, \quad \Delta x = \tfrac{1}{4}$

9 $\displaystyle\int_0^1 \sqrt{x^4 + 1} \, dx, \quad \Delta x = \tfrac{1}{4}$ 10 $\displaystyle\int_1^4 \sqrt{1 + 1/x} \, dx, \quad \Delta x = 0.5$

11 $\displaystyle\int_0^6 \frac{1}{x + 1} dx, \quad \Delta x = 1$ 12 $\displaystyle\int_0^{12} \frac{1}{2x + 3} dx, \quad \Delta x = 2$

13 $\displaystyle\int_1^{13} \frac{1}{x + \sqrt{x}} dx, \quad \Delta x = 3$ 14 $\displaystyle\int_0^4 \frac{1}{2 + \sqrt{x}} dx, \quad \Delta x = 1$

15 $\displaystyle\int_0^\pi \sin\theta\, d\theta, \quad \Delta x = \pi/2, \quad \pi/10$ **16** $\displaystyle\int_0^\pi \sin^2\theta\, d\theta, \quad \Delta x = \pi/2, \quad \pi/10$

17 $\displaystyle\int_0^1 e^x\, dx, \quad \Delta x = \tfrac{1}{4}$ **18** $\displaystyle\int_0^1 e^{x^2}\, dx, \quad \Delta x = \tfrac{1}{4}$

19 $\displaystyle\int_1^2 \ln x\, dx, \quad \Delta x = \tfrac{1}{4}$ **20** $\displaystyle\int_1^2 \ln(1/x)\, dx, \quad \Delta x = \tfrac{1}{4}$

☐ **21** Let f be continuous on the interval $[a, b]$ and let $\Delta x = (b - a)/n$ where n is a positive integer. Prove that the trapezoidal sum is equal to the Riemann sum plus $\tfrac{1}{2}(f(b) - f(a))\,\Delta x$, that is,

$$\sum_a^b \tfrac{1}{2}(f(x) + f(x + \Delta x))\,\Delta x = \left(\sum_a^b f(x)\,\Delta x\right) + \tfrac{1}{2}(f(b) - f(a))\,\Delta x.$$

Show that if $f(a) = f(b)$ then the trapezoidal sum and Riemann sum are equal.

☐ **22** Prove that for a linear function $f(x) = kx + c$, the trapezoidal sum is exactly equal to the integral.

☐ **23** Show that if $f(x)$ is concave downward, $f''(x) > 0$, then the trapezoidal sum is less than the definite integral of $f(x)$.

☐ **24** Show that for a quadratic function $f(x) = ax^2 + bx + c$, Simpson's approximation is equal to the definite integral.

☐ **25** Show that for a cubic function $f(x) = ax^3 + bx^2 + cx + d$, Simpson's approximation is still equal to the definite integral.

EXTRA PROBLEMS FOR CHAPTER 4

1 Evaluate $\sum_1^2 \dfrac{1}{x^2}\,\Delta x, \quad \Delta x = 1/4$

2 Evaluate $\sum_1^{10} \dfrac{1}{x^2}\,\Delta x, \quad \Delta x = 2$

3 Evaluate $\sum_{-3}^3 2^x\,\Delta x, \quad \Delta x = 1$

4 Evaluate $\sum_0^2 x\sqrt{x + 1}\,\Delta x, \quad \Delta x = 1/2$

5 If $F'(x) = 1/(2x - 1)^2$ for all $x \neq 1/2$, find $F(2) - F(1)$.

6 If $G'(t) = \sqrt{4t + 1}$ for all $t > -1/4$, find $G(2) - G(0)$.

7 A particle moves with velocity $v = (3 + 2\sqrt{t})^2$. How far does it move from times $t_0 = 1$ to $t_1 = 5$?

8 A particle moves with velocity $v = t^2\sqrt{t^3 - 1}$. How far does it move from times $t_0 = 1$ to $t_1 = 4$?

9 A particle moves with velocity $v = (t + 1)(2t + 3)$. If it has position $y_0 = 0$ at time $t = 0$, find its position at time $t = 10$.

10 A particle moves with acceleration $a = 1/t^4$. If it has velocity $v_0 = 4$ and position $y_0 = 2$ at time $t = 1$, find its position at time $t = 3$.

11 Find the area of the region under the curve $y = 1/\sqrt{x}, 1 \leq x \leq 4$.

12 Find the area of the region under the curve $y = \sqrt{x} - x\sqrt{x}, 0 \leq x \leq 1$.

In Problems 13–30, evaluate the integral.

13 $\displaystyle\int (1 - x)(2 + 3x)\, dx$ **14** $\displaystyle\int \left(2 + \frac{1}{x}\right)\left(2 - \frac{1}{x}\right) dx$

15 $\displaystyle\int \frac{x}{(x^2 - 1)^3}\, dx$

16 $\displaystyle\int (4x + 1)^{1/3}\, dx$

17 $\displaystyle\int (u/\sqrt{1 - 3u^2})\, du$

18 $\displaystyle\int x^{-2}\sqrt{2 + x^{-1}}\, dx$

19 $\displaystyle\int (\sqrt{2t + 1} - \sqrt{2t - 1})\, dt$

20 $\displaystyle\int \frac{2x + 1}{(x + 4)^3}\, dx$

21 $\displaystyle\int y\sqrt{y + 2}\, dy$

22 $\displaystyle\int (1 - \sqrt{x})^{-4}\, dx$

23 $\displaystyle\int \cos\left(\frac{x}{2}\right) dx$

24 $\displaystyle\int \sqrt{x}\sin\sqrt{x}\, dx$

25 $\displaystyle\int e^{-t}\, dt$

26 $\displaystyle\int \frac{t + 1}{t - 1}\, dt$

27 $\displaystyle\int_0^4 (y + \sqrt{y})\, dy$

28 $\displaystyle\int_2^6 (x/\sqrt{x^2 - 1})\, dx$

29 $\displaystyle\int_0^1 e^{4x}\, dx$

30 $\displaystyle\int_0^1 x \sin(x^2)\, dx$

31 Differentiate $\displaystyle\int_1^x \sqrt{t^3 + 2}\, dt$

32 Differentiate $\displaystyle\int_0^{3x} (t^2/(t^2 - 1))\, dt$

33 Differentiate $\displaystyle\int_u^4 \sqrt{x}\sqrt{x - 1}\, dx$

34 Differentiate $\displaystyle\int_y^{y^2} (1/(x + \sqrt{x}))\, dx$

35 Find the function F such that $F'(x) = x - 1$ for all x, and the minimum value of $F(x)$ is b.

36 Find the function F such that $F''(x) = x$ for all x, $F(0) = 1$, and $F(1) = 1$.

37 Find the function F such that $F''(x) = 6$ for all x, $F(x)$ has a minimum at $x = 1$, and the minimum value is 2.

38 Find all functions F such that $F''(x) = 1 + x^{-3}$ for all positive x.

☐ 39 Find the function F such that
$$F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$
and $F(0) = 1$.

☐ 40 Find the value of b such that the area of the region under the curve $y = x(b - x)$, $0 \leq x \leq b$, is 1.

☐ 41 Suppose f is increasing for $a \leq x \leq b$, and $\Delta x = (b - a)/n$ where n is a positive integer. Show that
$$\left| \sum_a^b f(x)\, \Delta x - \int_a^b f(x)\, dx \right| \leq [f(b) - f(a)]\, \Delta x$$

☐ 42 Suppose f is continuous for $a \leq x \leq b$. Show that
$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

☐ 43 Find the area of the top half of the ellipse $x^2/a^2 + y^2/b^2 = 1$ using the formula $\pi = 2\int_{-1}^1 \sqrt{1 - u^2}\, du$.

☐ 44 Evaluate $\int_{-1}^1 (1 - x)^{3/2}(1 + x)^{1/2}\, dx$ using the formula $\pi = 2\int_{-1}^1 \sqrt{1 - u^2}\, du$.

☐ 45 Find dy/dx if $y = \int_0^x x f(t)\, dt$.

☐ 46 Suppose $f(t)$ is continuous for all t and let $G(x) = \int_0^x (x - t) f(t)\, dt$. Prove that $G''(x) = f(x)$.

☐ 47 Prove that for any continuous functions f and g,

$$2AB \int_a^b f(x)g(x)\, dx \le A^2 \int_a^b f(x)^2\, dx + B^2 \int_a^b g(x)^2\, dx.$$

☐ **48**   Prove Schwartz' Inequality,

$$\int_a^b f(x)g(x)\, dx \le \sqrt{\int_a^b f(x)^2\, dx \int_a^b g(x)^2\, dx}.$$

Hint: Use the preceding problem.

☐ **49**   Suppose f is continuous and dx is positive infinitesimal. Show that

$$\sum_a^b f(x + \tfrac{1}{2}\, dx)\, dx \approx \int_a^b f(x)\, dx.$$

Hint: For each positive real c,

$$f(x) - c < f\left(x + \frac{1}{2}dx\right) < f(x) + c.$$

Use this to show that

$$\int_a^b f(x)\, dx - c(b - a) < \sum_a^b f\left(x + \frac{1}{2}dx\right) dx < \int_a^b f(x)\, dx + c(b - a).$$

☐ **50**   Suppose f is continuous, n is an integer, and dx is positive infinitesimal. Prove that

$$\sum_a^b f(x + n\, dx)\, dx \approx \int_a^b f(x)\, dx.$$