6

APPLICATIONS OF THE INTEGRAL

6.1 INFINITE SUM THEOREM

In Chapter 4 we obtained the formula

$$\text{Area} = \int_a^b f(x)\,dx$$

for the area of the region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = a$ and $x = b$.

In this chapter we shall obtain integral formulas for several other quantities arising in geometry and physics, such as volumes, curve lengths, and work. We begin with the Infinite Sum Theorem, which will be useful in justifying these formulas. It tells when a given function $B(a, b)$ is equal to the definite integral $\int_a^b h(x)\,dx$.

Any two infinitesimals are infinitely close to each other. The following definition helps us to keep track of how close to each other they are.

DEFINITION

Let ε, δ be infinitesimals and let Δx be a nonzero infinitesimal. We say that ε is ***infinitely close to δ compared to*** *Δx,*

$$\varepsilon \approx \delta \qquad \textit{(compared to } \Delta x\textit{)}, \quad \textit{if} \quad \varepsilon/\Delta x \approx \delta/\Delta x.$$

In Figure 6.1.1, an infinitesimal microscope within an infinitesimal microscope is used to show $\varepsilon \approx \delta$ (compared to Δx).

For example, $3\,\Delta x + 5\,\Delta x^2 \approx 3\,\Delta x - \Delta x^2 + \Delta x^3$ (compared to Δx)

but $3\,\Delta x + 5\,\Delta x^2 \approx 2\,\Delta x$ (compared to Δx).

The Infinite Sum Theorem is used when we have a quantity $B(u, w)$ depending on two variables $u < w$ in $[a, b]$, and the total value $B(a, b)$ is the sum of infinitesimal pieces

$$\Delta B = B(x, x + \Delta x).$$

The theorem gives a method of expressing $B(a, b)$ as a definite integral.

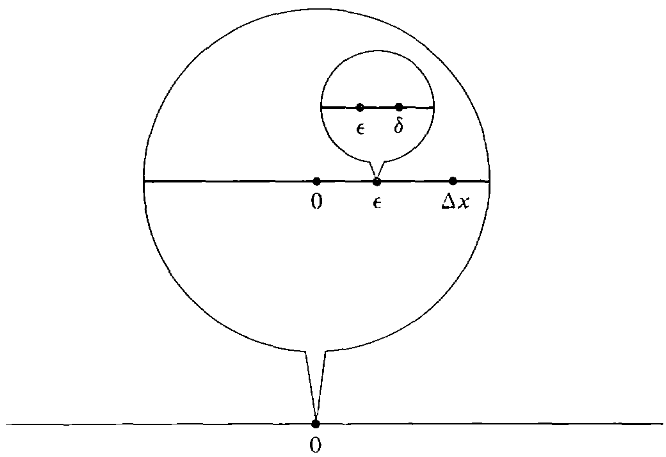


Figure 6.1.1 $\varepsilon \approx \delta$ (compared to Δx)

INFINITE SUM THEOREM

Let $B(u, w)$ be a real function of two variables that has the Addition Property in the interval $[a, b]$—i.e.,

$$B(u, w) = B(u, v) + B(v, w) \qquad \text{for } u < v < w \text{ in } [a, b].$$

Suppose $h(x)$ is a real function continuous on $[a, b]$ and for any infinitesimal subinterval $[x, x + \Delta x]$ of $[a, b]$,

$$\Delta B \approx h(x)\, \Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

Then $B(a, b)$ is equal to the integral

$$B(a, b) = \int_a^b h(x)\, dx.$$

Intuitively, the theorem says that if each infinitely small piece ΔB is infinitely close to $h(x)\,\Delta x$ compared to Δx, then the sum $B(a, b)$ of all these pieces is infinitely close to $\sum_a^b h(x)\,\Delta x$ (Figure 6.1.2). This is why we call it the Infinite Sum Theorem.

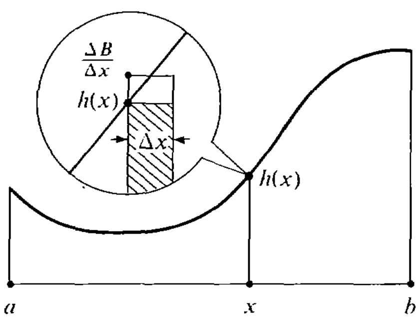


Figure 6.1.2

PROOF Divide the interval $[a, b]$ into subintervals of infinitesimal length Δx. Because $B(u, w)$ has the Addition Property, the sum of all the ΔB's is $B(a, b)$. Now let c be any positive real number. For each infinitesimal subinterval $[x, x + \Delta x]$ we have

$$h(x)\,\Delta x \approx \Delta B \qquad \text{(compared to } \Delta x)$$

$$h(x) \approx \frac{\Delta B}{\Delta x}$$

$$h(x) - c < \frac{\Delta B}{\Delta x} < h(x) + c$$

$$(h(x) - c)\,\Delta x < \Delta B < (h(x) + c)\,\Delta x.$$

Adding up, $$\sum_a^b (h(x) - c)\,\Delta x < B(a, b) < \sum_a^b (h(x) + c)\,\Delta x.$$

Now take standard parts,

$$\int_a^b (h(x) - c)\,dx \le B(a, b) \le \int_a^b (h(x) + c)\,dx$$

or $$\int_a^b h(x)\,dx - c(b - a) \le B(a, b) \le \int_a^b h(x)\,dx + c(b - a).$$

Since this holds for all positive real c, it follows that

$$B(a, b) = \int_a^b h(x)\,dx.$$

We shall use the Infinite Sum Theorem several times in this chapter. As a first illustration of the method, we derive again the formula from Chapter 4 for the area of the region between two curves, shown in Figure 6.1.3.

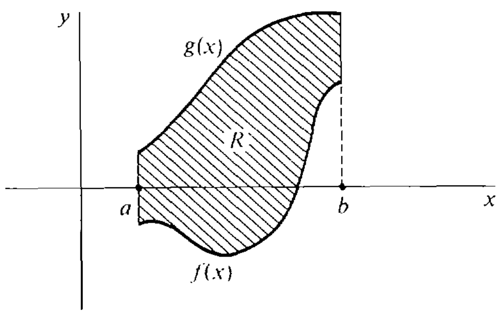


Figure 6.1.3

AREA BETWEEN TWO CURVES $\text{Area} = \int_a^b [g(x) - f(x)]\,dx.$

where f and g are continuous and $f(x) \le g(x)$ for $a \le x \le b$.

The justification of a definition resembles the proof of a theorem, but it shows that an intuitive concept is equivalent to a mathematical one. We shall now use the Infinitive Sum Theorem to give a justification of the formula for the area between two curves.

JUSTIFICATION We write $A(a, b)$ for the intuitive area of the region R between $f(x)$ and $g(x)$ from a to b. $A(u, w)$ has the Addition Property. Slice R into vertical strips of infinitesimal width Δx. Each strip is almost a rectangle of height $g(x) - f(x)$ and width Δx (Figure 6.1.4). The area $\Delta A = A(x, x + \Delta x)$ of the strip is infinitely close to the area of the rectangle compared to Δx,

$$\Delta A \approx [g(x) - f(x)]\, \Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

The infinite sum theorem now shows that $A(a, b)$ is the integral of $g(x) - f(x)$ from a to b.

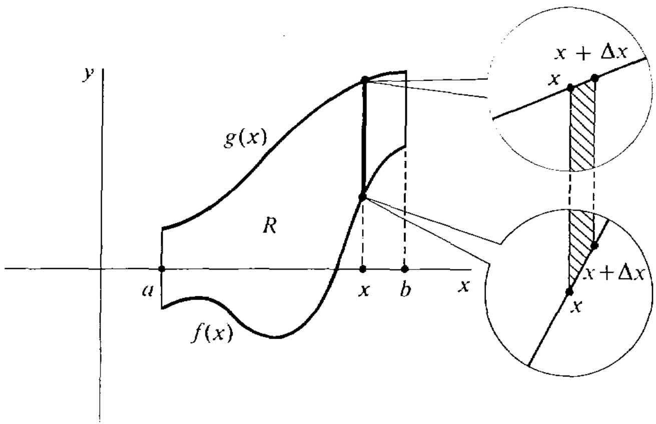


Figure 6.1.4

We now use the Infinite Sum Theorem to derive a formula for the volume of a solid when the area of each cross section is known. Suppose a solid S extends in the direction of the x-axis from $x = a$ to $x = b$, and for each x the plane perpendicular to the x-axis cuts the solid in a region of area $A(x)$, as shown in Figure 6.1.5. The area $A(x)$ is called the cross section of the solid at x. The volume is given by the formula:

VOLUME OF A SOLID $$V = \int_a^b A(x)\, dx.$$

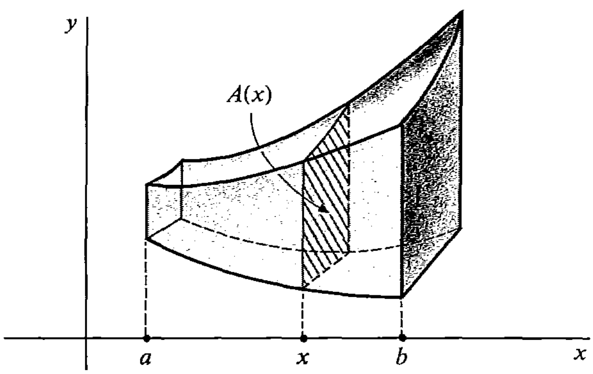


Figure 6.1.5

JUSTIFICATION Slice the solid S into vertical slabs of infinitesimal thickness Δx, as in Figure 6.1.6. Each slab, between x and $x + \Delta x$, has a face of area $A(x)$,

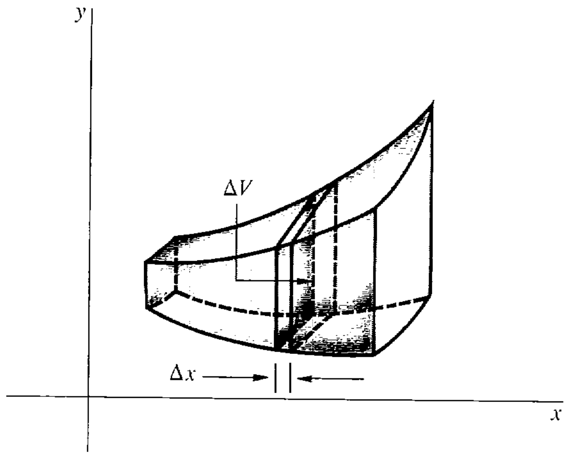


Figure 6.1.6

and thus its volume is given by

$$\Delta V \approx A(x)\,\Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

(The infinitesimal error arises because the area of the cross section changes slightly between x and $x + \Delta x$.) Then by the Infinite Sum Theorem,

$$V = \int_a^b A(x)\,dx.$$

The pattern used in justifying the two formulas in this section will be repeated again and again. First find a formula for an infinitesimal piece of volume ΔV. Then apply the Infinite Sum Theorem to get an integration formula for the total volume V.

EXAMPLE 1 Find the volume of a pyramid of height h whose base has area B, as in Figure 6.1.7.

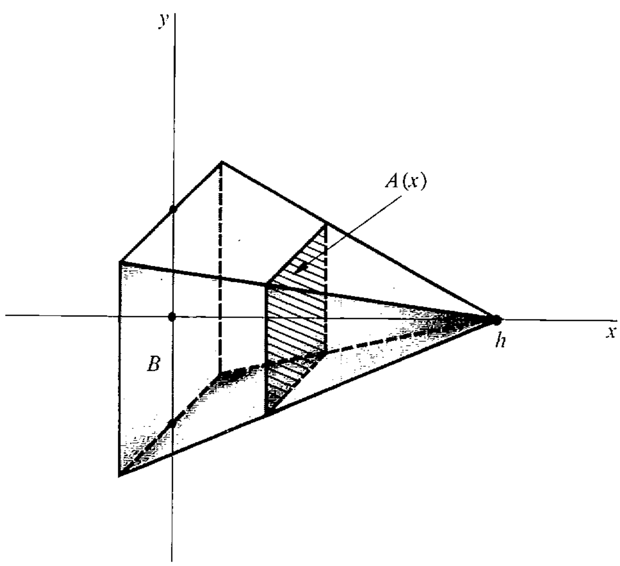


Figure 6.1.7 Example 1

Place the pyramid on its side with the apex at $x = 0$ and the base at $x = h$. We use the fact that at any point x between 0 and h, the cross section has area proportional to x^2, so that

$$\frac{A(x)}{x^2} = \frac{B}{h^2},$$

$$A(x) = \frac{Bx^2}{h^2}.$$

The volume is then

$$V = \int_0^h \frac{Bx^2}{h^2}\,dx = \frac{1}{3}\cdot\frac{Bx^3}{h^2}\Big]_0^h = \frac{1}{3}\cdot\frac{Bh^3}{h^2} = \frac{1}{3}Bh.$$

The solution is $V = (\frac{1}{3})Bh$.

EXAMPLE 2 A wedge is cut from a cylindrical tree trunk of radius 3 ft, by cutting the tree with two planes meeting on a line through the axis of the cylinder. The wedge is 1 ft thick at its thickest point. Find its volume.

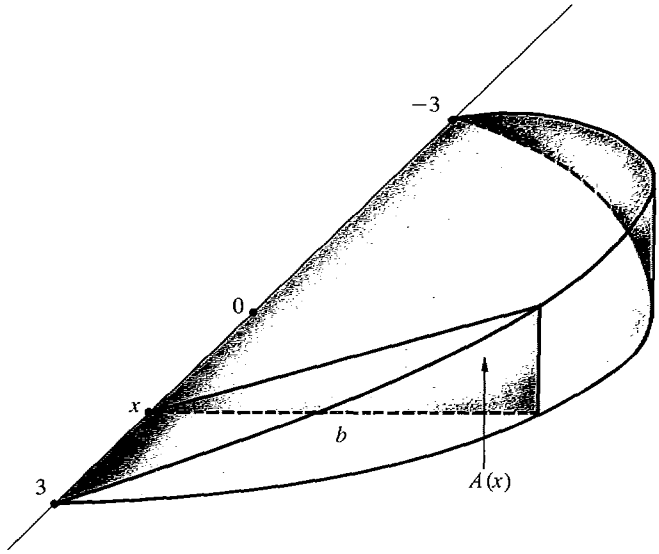


Figure 6.1.8 Example 2

The wedge is shown in Figure 6.1.8. The cross sections perpendicular to the x-axis are similar triangles. Place the edge along the x-axis with x from -3 to 3. At the thickest point, where $x = 0$, the cross section is a triangle with base 3 ft and altitude 1 ft. The base of the cross section triangle at x is

$$b = \sqrt{9 - x^2},$$

and the altitude is

$$\tfrac{1}{3}b = \tfrac{1}{3}\sqrt{9 - x^2}.$$

The area of the cross section is

$$A(x) = \tfrac{1}{2}\cdot \text{base}\cdot \text{altitude} = \tfrac{1}{2}b\cdot\tfrac{1}{3}b = \tfrac{1}{6}b^2 = \tfrac{1}{6}(9 - x^2).$$

The volume is thus

$$V = \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} \frac{1}{6}(9 - x^2)\,dx = \left(\frac{3}{2}x - \frac{1}{18}x^3\right)\Big]_{-3}^{3} = 6\ \text{ft}^3.$$

The solution is 6 cubic feet.

PROBLEMS FOR SECTION 6.1

1 The base of a solid is the triangle in the x, y-plane with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$. The cross sections perpendicular to the x-axis are squares with one side on the base. Find the volume of the solid.

2 The base of a solid is the region in the x, y-plane bounded by the parabola $y = x^2$ and the line $y = 1$. The cross sections perpendicular to the x-axis are squares with one side on the base. Find the volume of the solid.

3 Find the volume of the solid in Problem 1 if the cross sections are equilateral triangles with one side on the base.

4 Find the volume of the solid in Problem 2 if the cross sections are equilateral triangles with one side on the base.

5 Find the volume of the solid in Problem 1 if the cross sections are semicircles with diameter on the base.

6 Find the volume of the solid in Problem 2 if the cross sections are semicircles with diameter on the base.

7 Find the volume of a wedge cut from a circular cylinder of radius r by two planes whose line of intersection passes through the axis of the cylinder, if the wedge has thickness c at its thickest point.

8 Find the volume of the smaller wedge cut from a circular cylinder of radius r by two planes whose line of intersection is a chord at distance b from the axis of the cylinder, if the greatest thickness is c.

6.2 VOLUMES OF SOLIDS OF REVOLUTION

Integrals are used in this section to find the volume of a solid of revolution. A solid of revolution is generated by taking a region in the first quadrant of the plane and rotating it in space about the x- or y-axis (Figure 6.2.1).

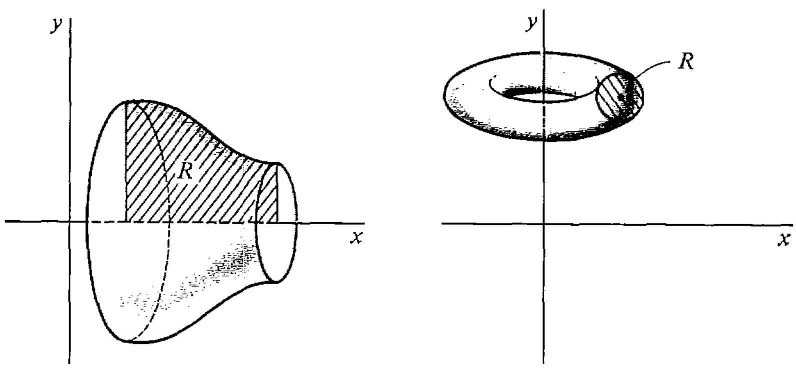


Figure 6.2.1 Solids of Revolution

We shall work with the region under a curve and the region between two curves. We use one method for rotating about the axis of the independent variable and another for rotating about the axis of the dependent variable.

For areas our starting point was the formula

$$\text{area} = \text{base} \times \text{height}$$

for the area of a rectangle. For volumes of a solid of revolution our starting point is the usual formula for the volume of a right circular cylinder (Figure 6.2.2).

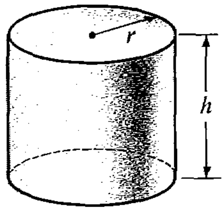


Figure 6.2.2

DEFINITION

*The **volume** of a right circular cylinder with height h and base of radius r is*

$$V = \pi r^2 h.$$

DISC METHOD: For rotations about the axis of the *independent variable.*

Let us first consider the region under a curve. Let R be the region under a curve $y = f(x)$ from $x = a$ to $x = b$, shown in Figure 6.2.3(a). x is the independent

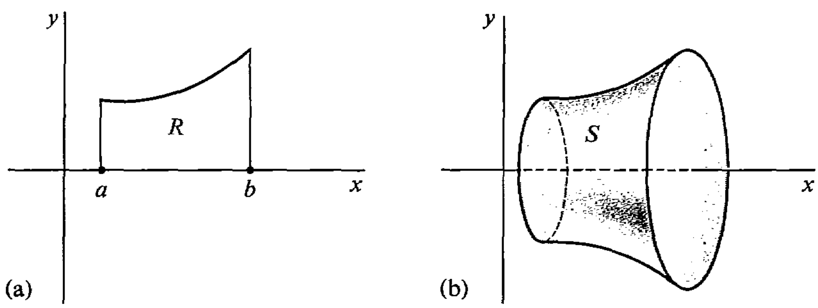


Figure 6.2.3

variable in this case. To keep R in the first quadrant we assume $0 \le a < b$ and $0 \le f(x)$. Rotate R about the x-axis, generating the solid of revolution S shown in Figure 6.2.3(b).

This volume is given by the formula below.

VOLUME BY DISC METHOD $\quad V = \int_a^b \pi(f(x))^2\,dx.$

To justify this formula we slice the region R into vertical strips of infinitesimal width Δx. This slices the solid S into discs of infinitesimal thickness Δx. Each disc is almost a cylinder of height Δx whose base is a circle of radius $f(x)$ (Figure 6.2.4). Therefore

$$\Delta V = \pi(f(x))^2\,\Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

Then by the Infinite Sum Theorem we get the desired formula

$$V = \int_a^b \pi(f(x))^2\,\Delta x.$$

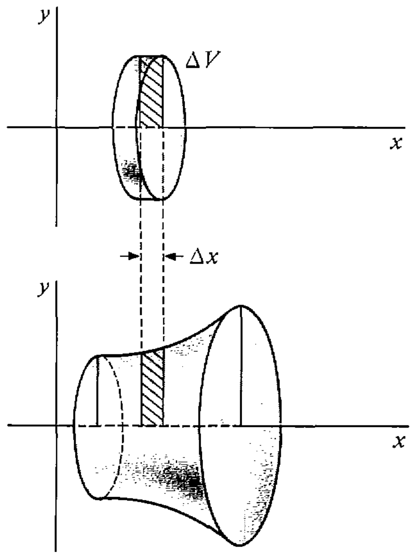


Figure 6.2.4 Disc Method

EXAMPLE 1 Find the volume of a right circular cone with height h and base of radius r.

It is convenient to center the cone on the x-axis with its vertex at the origin as shown in Figure 6.2.5. This cone is the solid generated by rotating about the x-axis the triangular region R under the line $y = (r/h)x, 0 \le x \le h$.

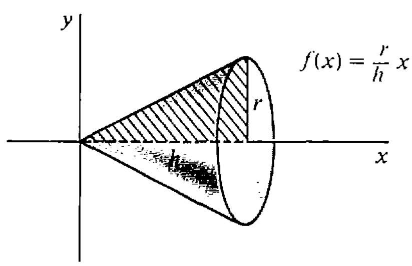


Figure 6.2.5

Since x is the independent variable we use the Disc Method. The volume formula gives

$$V = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \pi\frac{r^2}{h^2}\int_0^h x^2\,dx = \pi\frac{r^2}{h^2}\frac{x^3}{3}\Bigg]_0^h = \frac{1}{3}\pi r^2 h,$$

or

$$V = \tfrac{1}{3}\pi r^2 h.$$

Now we consider the region R between two curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$. Rotating R about the x-axis generates a solid of revolution S shown in Figure 6.2.6(c).

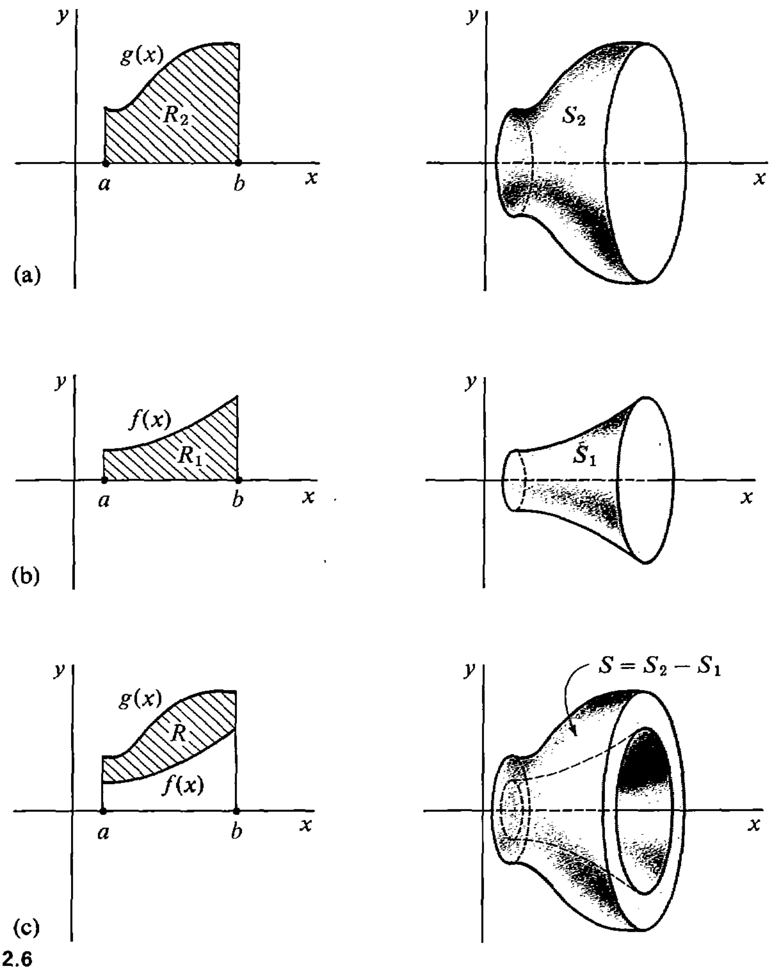


Figure 6.2.6

Let R_1 be the region under the curve $y = f(x)$ shown in Figure 6.2.6(b), and R_2, the region under the curve $y = g(x)$, shown in Figure 6.2.6(a). Then S can be found by removing the solid of revolution S_1 generated by R_1 from the solid of revolution S_2 generated by R_2. Therefore

$$\text{volume of } S = \text{volume of } S_2 - \text{volume of } S_1.$$

This justifies the formula

$$V = \int_a^b \pi(g(x))^2\,dx - \int_a^b \pi(f(x))^2\,dx.$$

We combine this into a single integral.

VOLUME BY DISC METHOD $V = \int_a^b \pi[(g(x))^2 - (f(x))^2]\,dx.$

Another way to see this formula is to divide the solid into annular discs (washers) with inner radius $f(x)$ and outer radius $g(x)$, as illustrated in Figure 6.2.7.

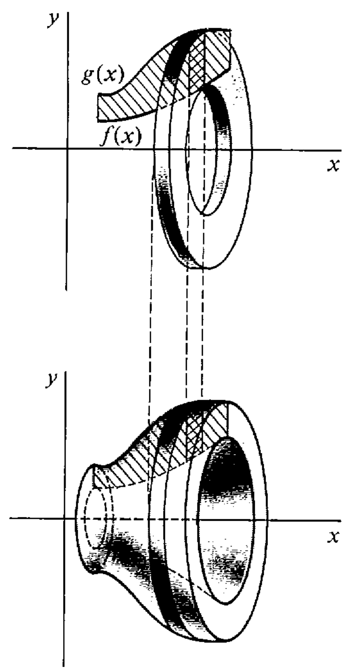


Figure 6.2.7

EXAMPLE 2 The region R between the curves $y = 2 - x^2$ and $y = x^2$ is rotated about the x-axis generating a solid S. Find the volume of S.

The curves $y = 2 - x^2$ and $y = x^2$ cross at $x = \pm 1$. The region is sketched in Figure 6.2.8. The volume is

$$\begin{aligned} V &= \int_{-1}^{1} \pi(2 - x^2)^2\,dx - \int_{-1}^{1} \pi(x^2)^2\,dx \\ &= \int_{-1}^{1} \pi(2 - x^2)^2 - \pi x^4\,dx \\ &= \int_{-1}^{1} \pi(4 - 4x^2)\,dx = \pi(4x - \tfrac{4}{3}x^3)\Big]_{-1}^{1} = 16\pi/3. \end{aligned}$$

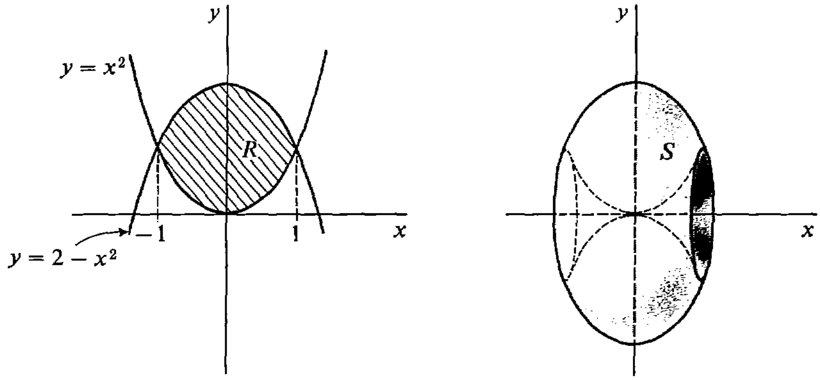


Figure 6.2.8

Warning: When using the disc method for a region between two curves, the correct formula is

$$V = \int_a^b \pi(g(x))^2\, dx - \int_a^b \pi(f(x))^2\, dx,$$

or

$$V = \int_a^b \pi[(g(x))^2 - (f(x))^2]\, dx.$$

A common mistake is to subtract $f(x)$ from $g(x)$ before squaring.

Wrong: $V = \int_a^b \pi(g(x) - f(x))^2\, dx.$

Wrong: (for Example 2):

$$V = \int_{-1}^1 \pi((2 - x^2) - x^2)^2\, dx = \int_{-1}^1 \pi(2 - 2x^2)^2\, dx$$

$$= \int_{-1}^1 \pi(4 - 8x^2 + 4x^4)\, dx = 64\pi/15.$$

CYLINDRICAL SHELL METHOD: For rotations about the axis of the *dependent variable.*

Let us again consider the region R under a curve $y = f(x)$ from $x = a$ to $x = b$, so that x is still the independent variable. This time rotate R about the y-axis to generate a solid of revolution S (Figure 6.2.9).

VOLUME BY CYLINDRICAL SHELL METHOD $\quad V = \int_a^b 2\pi x f(x)\, dx.$

Let us justify this formula. Divide R into vertical strips of infinitesimal width Δx as shown in Figure 6.2.10. When a vertical strip is rotated about the y-axis it generates a *cylindrical shell* of thickness Δx and volume ΔV. This cylindrical shell is the difference between an outer cylinder of radius $x + \Delta x$ and an inner cylinder of radius Δx. Both cylinders have height infinitely close to $f(x)$. Thus compared to Δx,

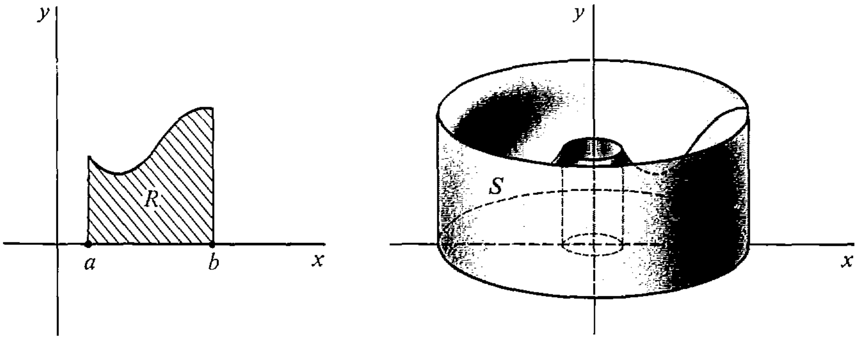


Figure 6.2.9

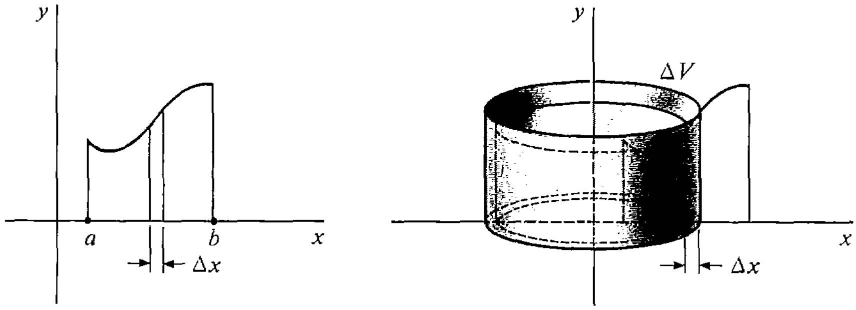


Figure 6.2.10 Cylindrical Shell Method

$$\begin{aligned}\Delta V &\approx \text{outer cylinder} - \text{inner cylinder}\\ &\approx \pi(x + \Delta x)^2 f(x) - \pi x^2 f(x)\\ &= \pi(x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2)f(x)\\ &= \pi(2x\,\Delta x + (\Delta x)^2)f(x) \approx \pi 2x\,\Delta x f(x),\end{aligned}$$

whence $\Delta V \approx 2\pi x f(x)\,\Delta x$ (compared to Δx).

By the Infinite Sum Theorem,

$$V = \int_a^b 2\pi x f(x)\,dx.$$

EXAMPLE 3 The region R between the line $y = 0$ and the curve $y = 2x - x^2$ is rotated about the y-axis to form a solid of revolution S. Find the volume of S.

We use the cylindrical shell method because y is the dependent variable. We see that the curve crosses the x-axis at $x = 0$ and $x = 2$, and sketch the region in Figure 6.2.11. The volume is

$$V = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 2x^2 - x^3\,dx = 2\pi(\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4)\Big]_0^2 = \tfrac{8}{3}\pi.$$

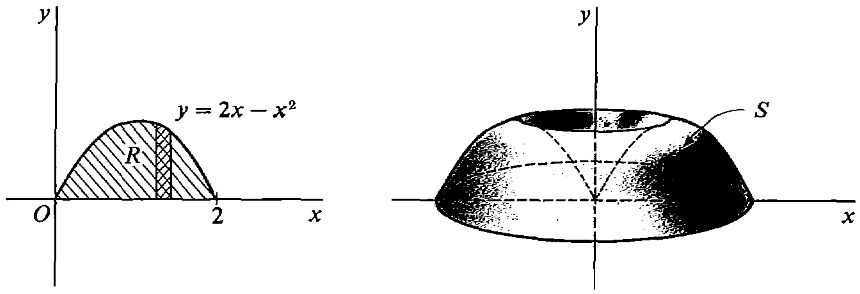


Figure 6.2.11

Now let R be the region between the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$, and generate the solid S by rotating R about the y-axis. The volume of S can be found by subtracting the volume of the solid S_1 generated by the region under $y = f(x)$ from the volume of the solid S_2 generated by the region under $y = g(x)$ (Figure 6.2.12). The formula for the volume is

$$V = S_2 - S_1 = \int_a^b 2\pi x g(x)\,dx - \int_a^b 2\pi x f(x)\,dx.$$

Combining into one integral, we get

VOLUME BY CYLINDRICAL SHELL METHOD $\quad V = \int_a^b 2\pi x(g(x) - f(x))\,dx.$

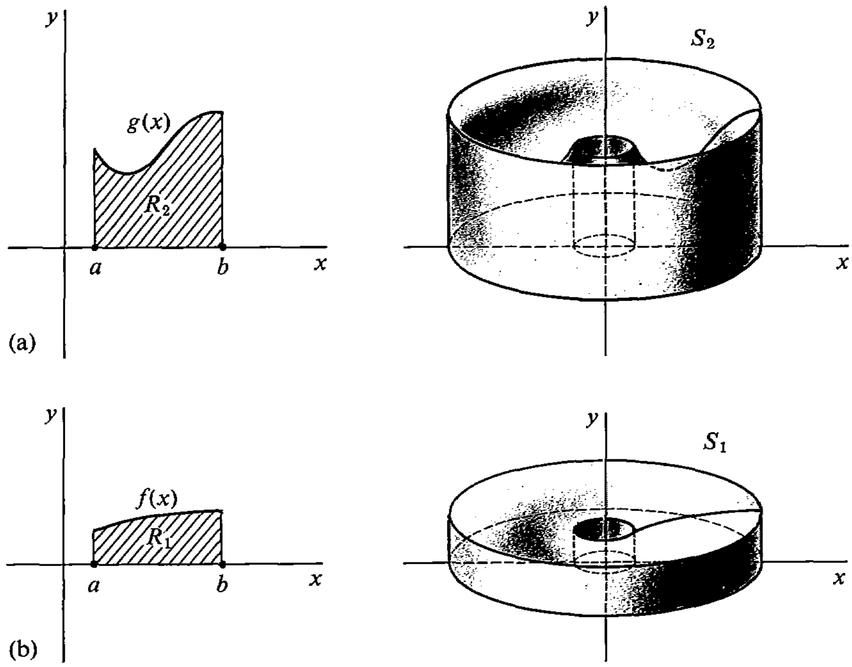


Figure 6.2.12

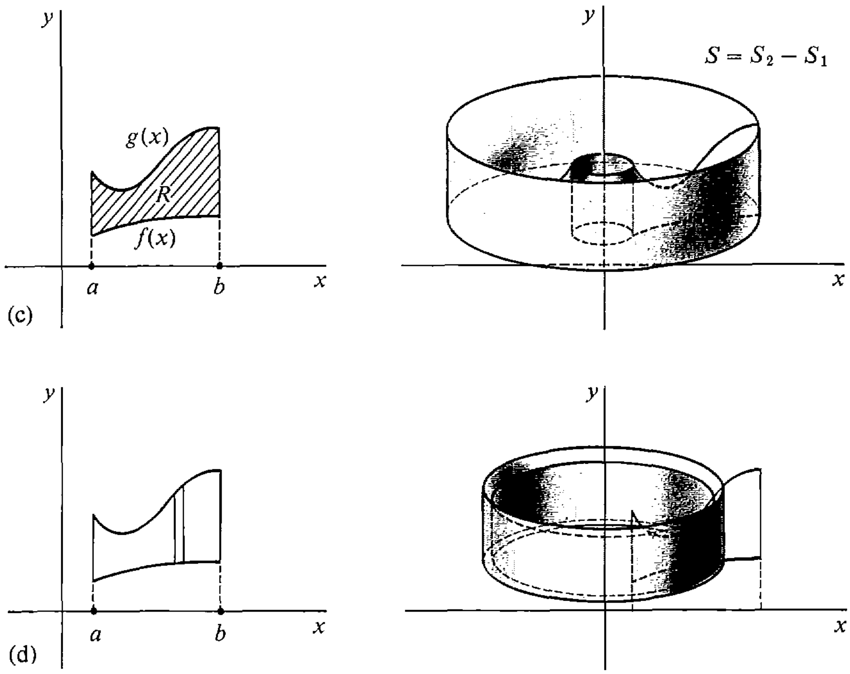


Figure 6.2.12

EXAMPLE 4 The region between the curves $y = x$ and $y = \sqrt{x}$ is rotated about the y-axis. Find the volume of the solid of revolution.

We make a sketch in Figure 6.2.13 and find that the curves cross at $x = 0$ and $x = 1$. We take x for the independent variable and use the Cylindrical Shell Method.

$$V = \int_0^1 2\pi x(\sqrt{x} - x)\,dx = \int_0^1 2\pi(x^{3/2} - x^2)\,dx = 2\pi(\tfrac{2}{5}x^{5/2} - \tfrac{1}{3}x^3)\Big]_0^1 = \tfrac{2}{15}\pi.$$

Some regions R are more easily described by taking y as the independent variable, so that R is the region between $x = f(y)$ and $x = g(y)$ for $c \le y \le d$. The volumes of the solids of revolution are then computed by integrating with respect to y. Often we have a choice of either x or y as the independent variable.

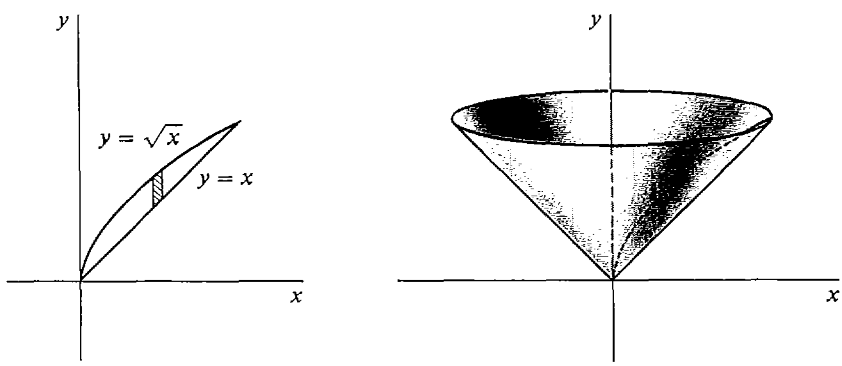


Figure 6.2.13

How can one decide whether to use the Disc or Cylindrical Shell Method? The answer depends on both the axis of rotation and the choice of independent variable. *Use the Disc Method when rotating about the axis of the independent variable. Use the Cylindrical Shell Method when rotating about the axis of the dependent variable.*

EXAMPLE 5 Derive the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere by both the Disc Method and the Cylindrical Shell Method.

The circle of radius r and center at the origin has the equation

$$x^2 + y^2 = r^2.$$

The region R inside this circle in the first quadrant will generate a hemisphere of radius r when it is rotated about the x-axis (Figure 6.2.14).

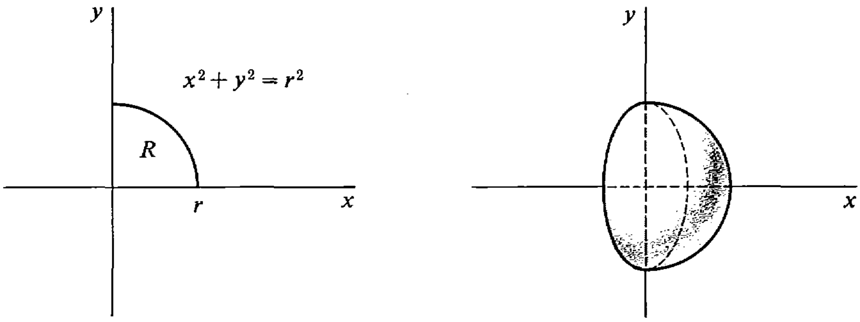


Figure 6.2.14

First take x as the independent variable and use the Disc Method. R is the region under the curve

$$y = \sqrt{r^2 - x^2}, \qquad 0 \le x \le r.$$

The hemisphere has volume

$$\begin{aligned} \tfrac{1}{2}V &= \int_0^r \pi(f(x))^2\,dx \\ &= \int_0^r \pi(r^2 - x^2)\,dx = \pi r^2 x - \tfrac{1}{3}\pi x^3\Big]_0^r \\ &= \pi r^3 - \tfrac{1}{3}\pi r^3 = \tfrac{2}{3}\pi r^3. \end{aligned}$$

Therefore the sphere has volume

$$V = \tfrac{4}{3}\pi r^3.$$

Now take y as the independent variable and use the Cylindrical Shell Method. R is the region under the curve

$$x = \sqrt{r^2 - y^2}, \qquad 0 \le y \le r.$$

The hemisphere has volume

$$\tfrac{1}{2}V = \int_0^r 2\pi y\sqrt{r^2 - y^2}\,dy.$$

Putting $u = r^2 - y^2$, $du = -2y\,dy$, we get

$$\tfrac{1}{2}V = \int_{r^2}^{0} 2\pi\sqrt{u}\left(-\tfrac{1}{2}\right)du = \int_{r^2}^{0} -\pi\sqrt{u}\,du = -\tfrac{2}{3}\pi u^{3/2}\Big]_{r^2}^{0} = \tfrac{2}{3}\pi r^3.$$

Thus again $V = \frac{4}{3}\pi r^3$.

PROBLEMS FOR SECTION 6.2

In Problems 1–10 the region under the given curve is rotated about (a) the x-axis, (b) the y-axis. Sketch the region and find the volumes of the two solids of revolution.

1 $y = x^2$, $0 \le x \le 1$

2 $y = x^3$, $0 \le x \le 1$

3 $y = \sqrt{x}$, $0 \le x \le 4$

4 $y = \sqrt{2x - 4}$, $2 \le x \le 4$

5 $y = 1 - x$, $0 \le x \le 1$

6 $y = x$, $1 \le x \le 2$

7 $y = \sqrt{1 + x^2}$, $0 \le x \le 1$

8 $y = \sqrt{x^2 - 4}$, $2 \le x \le 4$

9 $y = x^{-3}$, $1 \le x \le 2$

10 $y = 1/x$, $1 \le x \le 2$

In Problems 11–22 the region bounded by the given curves is rotated about (a) the x-axis, (b) the y-axis. Sketch the region and find the volumes of the two solids of revolution.

11 $x, y \ge 0$, $y = x^2\sqrt{1 - x^4}$

12 $y = 0$, $y = x - x^2$

13 $y = x$, $y = 2x$, $0 \le x \le 3$

14 $y = x^2$, $y = x$

15 $y = x^3$, $y = x^2$

16 $y = 3/x$, $y = 4 - x$

17 $x = 0$, $x = y - y^4$

18 $x = y$, $x = 2y - y^2$

19 $x = 0$, $x = y + 1/y$, $1 \le y \le 2$

20 $x \ge 0$, $y \ge 0$, $2x^2 + y^2 = 4$

21 $y = 0$, $y = x - 2$, $y = \sqrt{x}$

22 $y = \frac{3}{4}x$, $y = 1 - x$, $y = x - 1/x$ (first quadrant)

In Problems 23–34 the region under the given curve is rotated about the x-axis. Find the volume of the solid of revolution.

23 $y = \sqrt{\sin x}$, $0 \le x \le \pi$

24 $y = \cos x\sqrt{\sin x}$, $0 \le x \le \pi/2$

25 $y = \cos x - \sin x$, $0 \le x \le \pi/4$

26 $y = \sin(x/2) + \cos(x/2)$, $0 \le x \le \pi$

27 $y = e^x$, $0 \le x \le 1$

28 $y = e^{1-2x}$, $0 \le x \le 2$

29 $y = xe^{x^3}$, $0 \le x \le 1$

30 $y = \sqrt{e^x + 1}$, $0 \le x \le 3$

31 $y = 1/\sqrt{x}$, $1 \le x \le 2$

32 $y = \dfrac{1}{\sqrt{2x + 1}}$, $0 \le x \le 1$

33 $y = \sqrt{\dfrac{x - 1}{x}}$, $1 \le x \le 4$

34 $y = \sqrt{\dfrac{2x}{x + 1}}$, $0 \le x \le 1$

In Problems 35–46 the region is rotated about the x-axis. Find the volume of the solid of revolution.

35 $y = \dfrac{\sin x}{x}$, $\pi/2 \le x \le \pi$

36 $y = \dfrac{\cos x}{x}$, $\pi/6 \le x \le \pi/2$

37 $y = \sin(x^2)$, $0 \le x \le \sqrt{\pi}$

38 $y = \cos(x^2)$, $0 \le x \le \sqrt{\pi/2}$

39 $y = e^{x^2}$, $0 \le x \le 1$

40 $y = e^x/x$, $1 \le x \le 2$

41 $y = 1/xe^x$, $1 \le x \le 4$

42 $y = xe^{x^3}$, $1 \le x \le 2$

43 $y = x^{-2}, \quad 1 \le x \le 2$

44 $y = \dfrac{1}{x^2 + 1}, \quad 0 \le x \le 2$

45 $y = \dfrac{1}{2x^2 - 1}, \quad 1 \le x \le 2$

46 $y = \dfrac{\ln x}{x^2}, \quad 1 \le x \le 2$

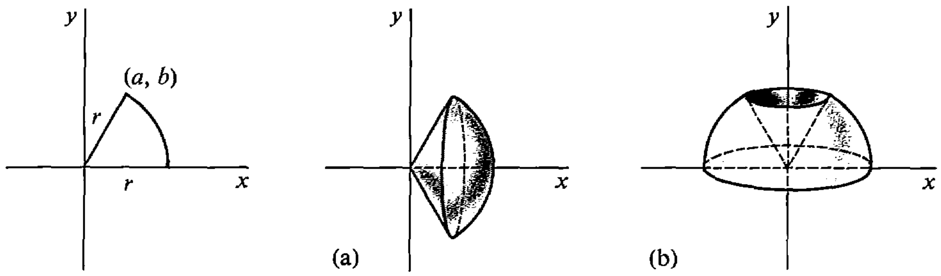
47 A hole of radius a is bored through the center of a sphere of radius r $(a < r)$. Find the volume of the remaining part of the sphere.

48 A sphere of radius r is cut by a horizontal plane at a distance c above the center of the sphere. Find the volume of the part of the sphere above the plane $(c < r)$.

49 A hole of radius a is bored along the axis of a cone of height h and base of radius r. Find the remaining volume $(a < r)$.

50 Find the volume of the solid generated by rotating an ellipse $a^2x^2 + b^2y^2 = 1$ about the x-axis. *Hint:* The portion of the ellipse in the first quadrant will generate half the volume.

51 The sector of a circle shown in the figure is rotated about (a) the x-axis, (b) the y-axis. Find the volumes of the solids of revolution.



52 The region bounded by the curves $y = x^2$, $y = x$ is rotated about (a) the line $y = -1$, (b) the line $x = -2$. Find the volumes of the solids of revolution.

53 Find the volume of the torus (donut) generated by rotating the circle of radius r with center at $(c, 0)$ around the y-axis $(r < c)$.

□ 54 (a) Find a general formula for the volume of the solid of revolution generated by rotating the region bounded by the curves $y = f(x)$, $y = g(x)$, $a \le x \le b$, about the line $y = -k$.

(b) Do the same for a rotation about the line $x = -h$.

6.3 LENGTH OF A CURVE

A segment of a curve in the plane (Figure 6.3.1) is described by

$$y = f(x), \qquad a \le x \le b.$$

What is its length? As usual, we shall give a definition and then justify it. A curve $y = f(x)$ is said to be *smooth* if its derivative $f'(x)$ is continuous. Our definition will assign a length to a segment of a smooth curve.

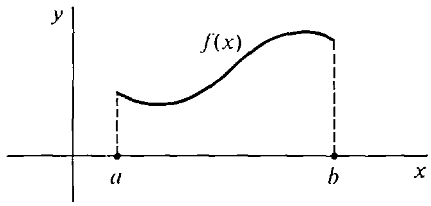


Figure 6.3.1

DEFINITION

Assume the function $y = f(x)$ has a continuous derivative for x in $[a, b]$, that is, the curve

$$y = f(x), \qquad a \le x \le b$$

*is smooth. The **length** of the curve is defined as*

$$s = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx.$$

Because $\sqrt{1 + (dy/dx)^2}\, dx = \sqrt{dx^2 + dy^2}$, the equation is sometimes written in the form

$$s = \int_a^b \sqrt{dx^2 + dy^2}$$

with the understanding that x is the independent variable. The length s is always greater than or equal to 0 because $a < b$ and

$$\sqrt{1 + (dy/dx)^2} > 0.$$

JUSTIFICATION Let $s(u, w)$ be the intuitive length of the curve between $t = u$ and $t = w$. The function $s(u, w)$ has the Addition Property; the length of the curve from u to w equals the length from u to v plus the length from v to w. Figure 6.3.2 shows an infinitesimal piece of the curve from x to $x + \Delta x$. Its length is $\Delta s = s(x, x + \Delta x)$.

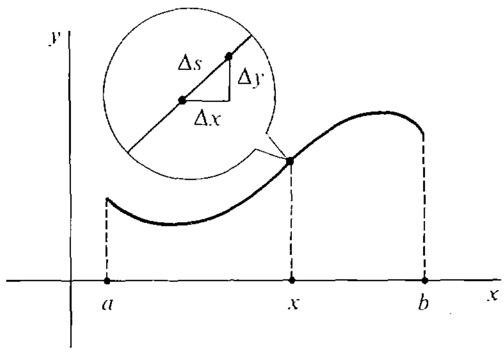


Figure 6.3.2

The slope dy/dx is a continuous function of x, and therefore changes only by an infinitesimal amount between x and $x + \Delta x$. Thus the infinitesimal piece of the curve is almost a straight line, the hypotenuse of a right triangle with sides Δx and Δy. Hence

$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2} \qquad \text{(compared to } \Delta x\text{)}.$$

Dividing by Δx,

$$\frac{\Delta s}{\Delta x} \approx \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} = \sqrt{\left(\frac{\Delta x}{\Delta x}\right)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2} \approx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Then $\Delta s \approx \sqrt{1 + (dy/dx)^2}\,\Delta x$ (compared to Δx).

Using the Infinite Sum Theorem,

$$s(a, b) = \int_a^b \sqrt{1 + (dy/dx)^2}\,dx.$$

EXAMPLE 1 Find the length of the curve

$$y = 2x^{3/2}, \qquad 0 \le x \le 1$$

shown in Figure 6.3.3. We have

$$dy/dx = 3x^{1/2}, \qquad s = \int_0^1 \sqrt{1 + 9x}\,dx.$$

Put $u = 1 + 9x$. Then

$$s = \int_1^{10} \tfrac{1}{9}\sqrt{u}\,du = \tfrac{2}{3}\cdot\tfrac{1}{9}u^{3/2}\Big]_1^{10} = \tfrac{2}{27}(\sqrt{1000} - 1).$$

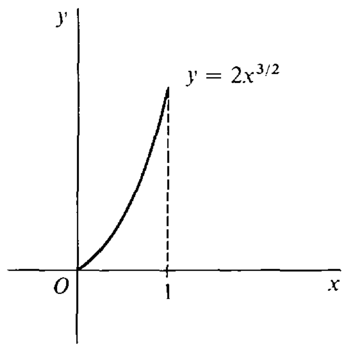


Figure 6.3.3

Sometimes a curve in the (x, y) plane is given by parametric equations

$$x = f(t), \qquad y = g(t), \qquad c \le t \le d.$$

A natural example is the path of a moving particle where t is time. We give a formula for the length of such a curve.

DEFINITION

Suppose the functions

$$x = f(t), \qquad y = g(t)$$

have continuous derivatives and the parametric curve does not retrace its path for t in $[a, b]$. The length of the curve is defined by

$$s = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt.$$

JUSTIFICATION The infinitesimal piece of the curve (Figure 6.3.4) from t to $t + \Delta t$

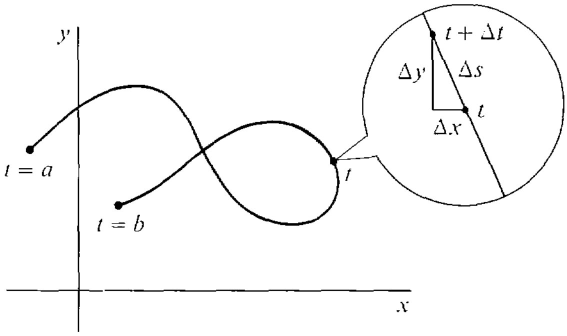


Figure 6.3.4

is almost a straight line, so its length Δs is given by

$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2} \qquad \text{(compared to } \Delta t\text{)},$$

$$\Delta s \approx \sqrt{(dx/dt)^2 + (dy/dt)^2}\, \Delta t \qquad \text{(compared to } \Delta t\text{)}.$$

By the Infinite Sum Theorem,

$$s = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$

The general formula for the length of a parametric curve reduces to our first formula when the curve is given by a simple equation $x = g(y)$ or $y = f(x)$.

If $y = f(x)$, $a \leq x \leq b$, we take $x = t$ and get

$$s = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx.$$

If $x = g(y)$, $a \leq y \leq b$, we take $y = t$ and get

$$s = \int_a^b \sqrt{(dx/dy)^2 + 1}\, dy.$$

EXAMPLE 2 Find the length of the path of a ball whose motion is given by

$$x = 20t, \qquad y = 32t - 16t^2$$

from $t = 0$ until the ball hits the ground. (Ground level is $y = 0$, see Figure 6.3.5.) The ball is at ground level when

$$32t - 16t^2 = 0, \qquad t = 0 \quad \text{and} \quad t = 2.$$

We have

$$dx/dt = 20, \qquad dy/dt = 32 - 32t,$$

$$s = \int_0^2 \sqrt{20^2 + (32 - 32t)^2}\, dt.$$

We cannot evaluate this integral yet, so the answer is left in the above form. We can get an approximate answer by the Trapezoidal Rule. When $\Delta x = \frac{1}{5}$, the Trapezoidal Approximation is

$$s \sim 53.5 \qquad \text{error} \leq 0.4.$$

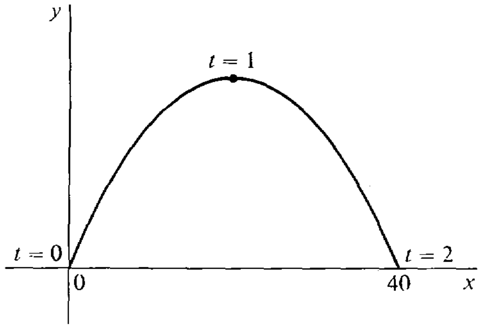


Figure 6.3.5

The following example shows what happens when a parametric curve does retrace its path.

EXAMPLE 3 Let

$$x = 1 - t^2, \qquad y = 1, \qquad -1 \leq t \leq 1.$$

As t goes from -1 to 1, the point (x, y) moves from (0, 1) to (1, 1) and then back along the same line to (0, 1) again. The path is shown in Figure 6.3.6.

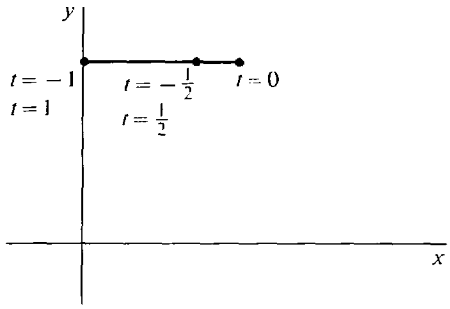


Figure 6.3.6

The path has length one. However, the point goes along the path twice for a total distance of two. The length formula gives the total distance the point moves.

$$s = \int_{-1}^{1} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-1}^{1} \sqrt{(-2t)^2 + 0^2}\, dt$$
$$= \int_{-1}^{1} \sqrt{4t^2}\, dt = \int_{-1}^{1} 2|t|\, dt = 2.$$

We next prove a theorem which shows the connection between the length of an arc and the area of a sector of a circle. Given two points P and Q on a circle with center O, the *arc PQ* is the portion of the circle traced out by a point moving from P to Q in a counterclockwise direction. The *sector POQ* is the region bounded by the arc PQ and the radii OP and OQ as shown in Figure 6.3.7.

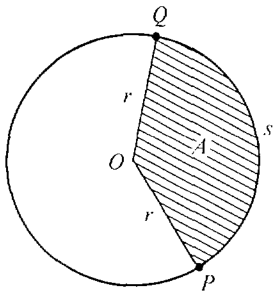


Figure 6.3.7

THEOREM

Let P and Q be two points on a circle with center O. The area A of the sector POQ is equal to one half the radius r times the length s of the arc PQ,

$$A = \tfrac{1}{2}rs.$$

DISCUSSION The theorem is intuitively plausible because if we consider an infinitely small arc Δs of the circle as in Figure 6.3.8, then the corresponding sector is almost a triangle of height r and base Δs, so it has area

$$\Delta A \approx \tfrac{1}{2}r\,\Delta s \qquad \text{(compared to } \Delta s\text{)}.$$

Summing up, we expect that $A = \frac{1}{2}rs$.

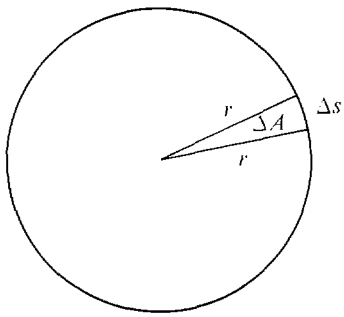


Figure 6.3.8

We can derive the formula $C = 2\pi r$ for the circumference of a circle using the theorem. By definition, π is the area of a circle of radius one,

$$\pi = \int_{-1}^{1} 2\sqrt{1 - y^2}\,dy.$$

Then a circle of radius r has area

$$A = \int_{-r}^{r} 2\sqrt{r^2 - y^2}\,dy = \int_{-1}^{1} 2r^2\sqrt{1 - (y/r)^2}\,d(y/r) = \pi r^2.$$

Therefore the circumference C is given by

$$A = \tfrac{1}{2}rC, \qquad \pi r^2 = \tfrac{1}{2}rC, \qquad C = 2\pi r.$$

PROOF OF THEOREM To simplify notation assume that the center O is at the origin, P is the point $(0, r)$ on the x-axis, and Q is a point (x, y) which varies along the circle (Figure 6.3.9). We may take y as the independent variable and

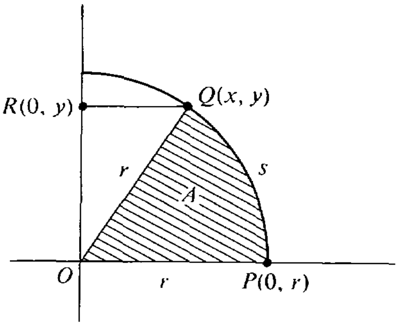


Figure 6.3.9

use the equation $x = \sqrt{r^2 - y^2}$ for the right half of the circle. Then A and s depend on y. Our plan is to show that

$$\frac{dA}{dy} = \frac{1}{2} r \frac{ds}{dy}.$$

First, we find dx/dy:

$$\frac{dx}{dy} = \frac{-y}{\sqrt{r^2 - y^2}} = -\frac{y}{x}.$$

Using the definition of arc length,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{y^2}{x^2}} = \sqrt{\frac{x^2 + y^2}{x^2}} = \frac{r}{x}.$$

The triangle OQR in the figure has area $\frac{1}{2}xy$, so the sector has area

$$A = \int_0^y x\,dy - \frac{1}{2}xy.$$

Then $$\frac{dA}{dy} = x - \frac{1}{2}\left(x + y\frac{dx}{dy}\right) = \frac{1}{2}x - \frac{1}{2}y\left(-\frac{y}{x}\right) = \frac{1}{2}\left(x + \frac{y^2}{x}\right)$$
$$= \frac{1}{2}\left(\frac{x^2 + y^2}{x}\right) = \frac{1}{2}\frac{r^2}{x}.$$

Thus $$\frac{dA}{dy} = \frac{1}{2}\frac{r^2}{x}, \qquad \frac{ds}{dy} = \frac{r}{x}, \qquad \frac{dA}{dy} = \frac{1}{2}r\frac{ds}{dy}.$$

So A and $\frac{1}{2}rs$ differ and only by a constant. But when $y = 0$, $A = \frac{1}{2}rs = 0$. Therefore $A = \frac{1}{2}rs$.

To prove the formula $A = \frac{1}{2}rs$ for arcs which are not within a single quadrant we simply cut the arc into four pieces each of which 'is within a single quadrant.

PROBLEMS FOR SECTION 6.3

Find the lengths of the following curves.

1 $y = \frac{2}{3}(x+2)^{3/2}, \quad 0 \le x \le 3$

2 $y = (x^2 + \frac{2}{3})^{3/2}, \quad -2 \le x \le 5$

3 $(3y-1)^2 = x^3, \quad 0 \le x \le 2$

4 $y = (4/5)x^{5/4}, \quad 0 \le x \le 1$

5 $y = (x-1)^{2/3}, \quad 1 \le x \le 9$ *Hint:* Solve for x as a function of y.

6 $y = \dfrac{x^3}{12} + \dfrac{1}{x}, \quad 1 \le x \le 3$

7 $x = \dfrac{y^4 + 3}{6y}, \quad 3 \le y \le 6$

8 $y = \frac{1}{3}x\sqrt{x} - \sqrt{x}, \quad 1 \le x \le 100$

9 $y = \frac{3}{5}x^{5/3} - \frac{3}{4}x^{1/3}, \quad 1 \le x \le 8$

10 $8x = 2y^4 + y^{-2}, \quad 1 \le y \le 2$

11 $x^{2/3} + y^{2/3} = 1$, first quadrant

12 $y = \int_0^x \sqrt{t^2 + 2t}\,dt, \quad 0 \le x \le 10$

13 $y = 2\int_1^x \sqrt{t^2 + t}\,dt, \quad 2 \le x \le 6$

14 $y = \int_1^{2x} \sqrt{t^{-4} + t^{-2}}\,dt, \quad 1 \le x \le 3$

15 $x = \int_1^y \sqrt{\sqrt{t} - 1}\,dt, \quad 1 \le y \le 4$

16 $y = \int_0^{x^2} (\sqrt{t} + 1)^{-2}\,dt, \quad 0 \le x \le 1$

17 Find the distance travelled from $t = 0$ to $t = 1$ by an object whose motion is $x = t^{3/2}$, $y = (3-t)^{3/2}$.

18 Find the distance moved from $t = 0$ to $t = 1$ by a particle whose motion is given by $x = 4(1-t)^{3/2}$, $y = 2t^{3/2}$.

19 Find the distance travelled from $t = 1$ to $t = 4$ by an object whose motion is given by $x = t^{3/2}$, $y = 9t$.

20 Find the distance travelled from time $t = 0$ to $t = 3$ by a particle whose motion is given by the parametric equations $x = 5t^2$, $y = t^3$.

21 Find the distance moved from $t = 0$ to $t = 2\pi$ by an object whose motion is $x = \cos t$, $y = \sin t$.

22 Find the distance moved from $t = 0$ to $t = \pi$ by an object with motion $x = 3\cos 2t$, $y = 3\sin 2t$.

23 Find the distance moved from $t = 0$ to $t = 2\pi$ by an object with motion $x = \cos^2 t$, $y = \sin^2 t$.

24 Find the distance moved by an object with motion $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le 1$.

25 Let $A(t)$ and $L(t)$ be the area under the curve $y = x^2$ from $x = 0$ to $x = t$, and the length of the curve from $x = 0$ to $x = t$, respectively. Find $d(A(t))/d(L(t))$.

In Problems 26–30, find definite integrals for the lengths of the curves, but do not evaluate the integrals.

26 $y = x^3, \quad 0 \le x \le 1$

27 $y = 2x^2 - x + 1, \quad 0 \le x \le 4$

28 $x = 1/t, \quad y = t^2, \quad 1 \le t \le 5$

29 $x = 2t + 1, \quad y = \sqrt{t}, \quad 1 \le t \le 2$

30 The circumference of the ellipse $x^2 + 4y^2 = 1$.

31 Set up an integral for the length of the curve $y = \sqrt{x}$, $1 \le x \le 2$, and find the Trapezoidal Approximation where $\Delta x = \frac{1}{4}$.

32 Set up an integral for the length of the curve $x = t^2 - t$, $y = \frac{4}{3}t^{3/2}$, $0 \le t \le 1$, and find the Trapezoidal Approximation where $\Delta t = \frac{1}{4}$.

33 Set up an integral for the length of the curve $y = 1/x$, $1 \le x \le 5$, and find the Trapezoidal Approximation where $\Delta x = 1$.

34 Set up an integral for the length of the curve $y = x^2$, $-1 \le x \le 1$, and find the Trapezoidal Approximation where $\Delta x = \frac{1}{2}$.

□ 35 Suppose the same curve is given in two ways, by a simple equation $y = F(x)$, $a \le x \le b$ and by parametric equations $x = f(t)$, $y = g(t)$, $c \le t \le d$. Assuming all derivatives are continuous and the parametric curve does not retrace its path, prove that the two formulas for curve length give the same values. *Hint:* Use integration by change of variables.

6.4 AREA OF A SURFACE OF REVOLUTION

When a curve in the plane is rotated about the x- or y-axis it forms a *surface of revolution*, as in Figure 6.4.1.

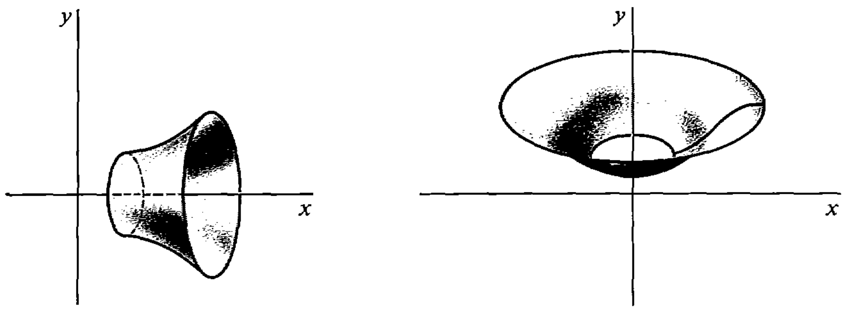


Figure 6.4.1 Surfaces of Revolution

The simplest surfaces of revolution are the right circular cylinders and cones. We can find their areas without calculus.

Figure 6.4.2 shows a right circular cylinder with height h and base of radius r. When the lateral surface is slit vertically and opened up it forms a rectangle with height h and base $2\pi r$. Therefore its area is

$$\textit{lateral area of cylinder} = 2\pi hr.$$

Figure 6.4.3 shows a right circular cone with slant height l and base of radius r.

When the cone is slit vertically and opened up, it forms a circular sector with radius l and arc length $s = 2\pi r$. Using the formula $A = \frac{1}{2} sl$ for the area of a sector, we see that the lateral surface of the cone has area

$$\textit{lateral area of cone} = \pi rl.$$

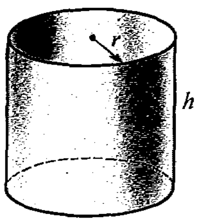


Figure 6.4.2

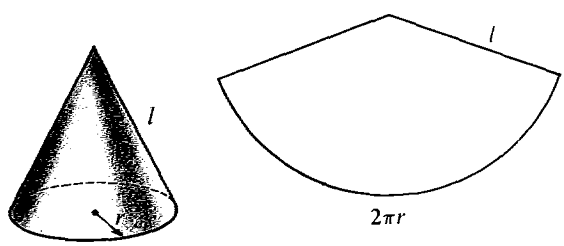


Figure 6.4.3

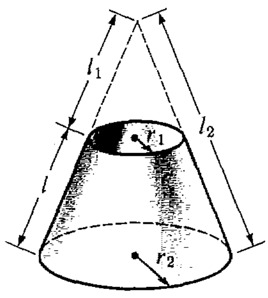


Figure 6.4.4 Cone frustum

Figure 6.4.4 shows the frustum of a cone with smaller radius r_1, larger radius r_2, and slant height l. The formula for the area of the lateral surface of a frustum of a cone is

$$\textit{lateral area of frustum} = \pi(r_1 + r_2)l.$$

This formula is justified as follows. The frustum is formed by removing a cone of radius r_1 and slant height l_1 from a cone of radius r_2 and slant height l_2. The frustum therefore has lateral area

$$A = \pi r_2 l_2 - \pi r_1 l_1.$$

The slant heights are proportional to the radii,

$$\frac{l_1}{r_1} = \frac{l_2}{r_2}, \qquad \text{so} \quad r_1 l_2 = r_2 l_1.$$

The slant height l of the frustum is

$$l = l_2 - l_1.$$

Using the last two equations,

$$\begin{aligned} \pi(r_1 + r_2)l &= \pi(r_2 + r_1)(l_2 - l_1) \\ &= \pi(r_2 l_2 + r_1 l_2 - r_2 l_1 - r_1 l_1) \\ &= \pi r_2 l_2 - \pi r_1 l_1 = A. \end{aligned}$$

A surface of revolution can be sliced into frustums in the same way that a solid of revolution can be sliced into discs or cylindrical shells. Consider a smooth curve segment

$$y = f(x), \qquad a \le x \le b$$

in the first quadrant. When this curve segment is rotated about the y-axis it forms a surface of revolution (Figure 6.4.5).

Here is the formula for the area.

AREA OF SURFACE OF REVOLUTION

$$A = \int_a^b 2\pi x\sqrt{1 + (dy/dx)^2}\, dx \qquad (\textit{rotating about } y\textit{-axis}).$$

To justify this formula we begin by dividing the interval $[a, b]$ into infinitesimal subintervals of length Δx. This divides the curve into pieces of infinitesi-

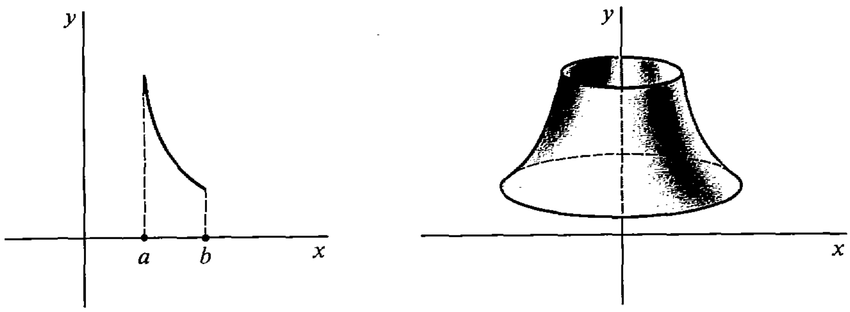


Figure 6.4.5

mal length Δs. When a piece Δs of the curve is rotated about the x-axis it sweeps out a piece of the surface, ΔA (Figure 6.4.6). Since Δs is almost a line segment, ΔA is almost a cone frustum of slant height Δs, and bases of radius x and $x + \Delta x$. Thus compared to Δx,

$$\Delta s \approx \sqrt{1 + (dy/dx)^2}\,\Delta x,$$
$$\Delta A \approx \pi(x + (x + \Delta x))\,\Delta s \approx 2\pi x\,\Delta s,$$
$$\Delta A \approx 2\pi x\sqrt{1 + (dy/dx)^2}\,\Delta x.$$

Then by the Infinite Sum Theorem,

$$A = \int_a^b 2\pi x\sqrt{1 + (dy/dx)^2}\,dx.$$

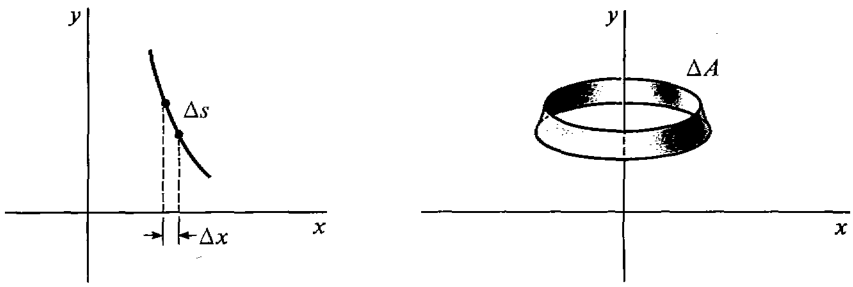


Figure 6.4.6

EXAMPLE 1 The line segment $y = 3x$, from $x = 1$ to $x = 4$, is rotated about the y-axis (Figure 6.4.7). Find the area of the surface of revolution.

FIRST SOLUTION We use the integration formula. $dy/dx = 3$, so

$$A = \int_1^4 2\pi x\sqrt{1 + (dy/dx)^2}\,dx$$
$$= \int_1^4 2\pi x\sqrt{1 + 3^2}\,dx = 2\pi\sqrt{10}\int_1^4 x\,dx$$
$$= 2\pi\sqrt{10}\,\frac{x^2}{2}\bigg]_1^4 = 2\pi\sqrt{10}\left(\frac{16 - 1}{2}\right) = 15\pi\sqrt{10}.$$

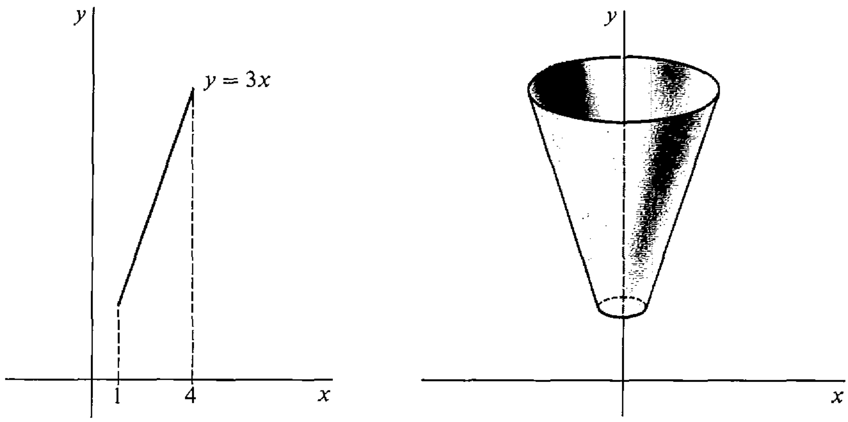


Figure 6.4.7

SECOND SOLUTION This surface of revolution is a frustum of a cone, so the formula for the lateral area of a frustum can be used directly. From the diagram we see that the radii and slant height are:

$$r_1 = 1, \qquad r_2 = 4,$$
$$l = \text{distance from } (1, 3) \text{ to } (4, 12)$$
$$= \sqrt{(4-1)^2 + (12-3)^2} = \sqrt{3^2 + 9^2} = \sqrt{90} = 3\sqrt{10}.$$

Then $\quad A = \pi(r_1 + r_2)l = \pi(1+4)3\sqrt{10} = 15\pi\sqrt{10}.$

EXAMPLE 2 The curve $y = \frac{1}{2}x^2$, $0 \le x \le 1$, is rotated about the y-axis (Figure 6.4.8). Find the area of the surface of revolution.

$$\frac{dy}{dx} = x,$$

$$A = \int_0^1 2\pi x\sqrt{1 + (dy/dx)^2}\,dx$$
$$= \int_0^1 2\pi x\sqrt{1 + x^2}\,dx = \int_1^2 \pi\sqrt{u}\,du \qquad (\text{where } u = 1 + x^2)$$
$$= \tfrac{2}{3}\pi u^{3/2}\Big]_1^2 = \tfrac{2}{3}\pi(2\sqrt{2} - 1).$$

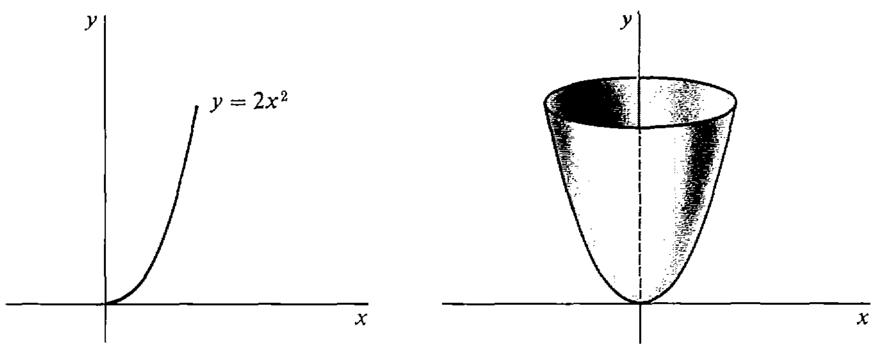


Figure 6.4.8

In finding a formula for surface area, why did we divide the surface into frustums of cones instead of into cylinders (as we did for volumes)? The reason is that to use the Infinite Sum Theorem we need something which is infinitely close to a small piece ΔA of area *compared to* Δx. The small frustum has area

$$(2x + \Delta x)\pi \, \Delta s$$

which is infinitely close to ΔA compared to Δx because it almost has the same shape as ΔA (Figure 6.4.9). The small cylinder has area $2x\pi \, \Delta y$. While this area is infinitesimal, it is not infinitely close to ΔA compared to Δx, because on dividing by Δx we get

$$\frac{\text{area of frustum}}{\Delta x} = 2x\pi \frac{\Delta s}{\Delta x} + \pi \, \Delta x \frac{\Delta s}{\Delta x} \approx 2x\pi \frac{ds}{dx},$$

$$\frac{\text{area of cylinder}}{\Delta x} = 2x\pi \frac{\Delta y}{\Delta x} \approx 2x\pi \frac{dy}{dx}.$$

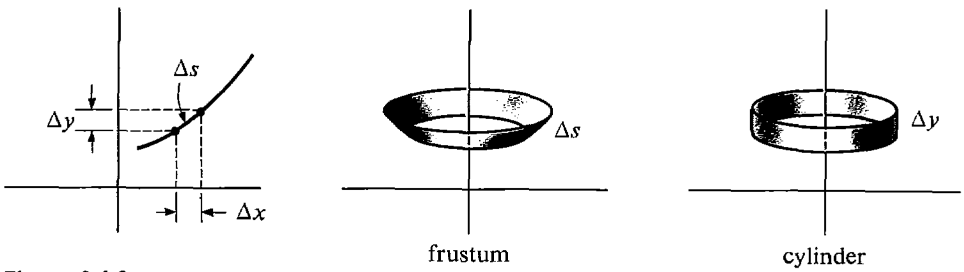


Figure 6.4.9

Approximating the surface by small cylinders would give us the different and incorrect value $\int_a^b 2\pi x \frac{dy}{dx} \, dx$ for the surface area.

When a curve is given by parametric equations we get a formula for surface area of revolution analogous to the formula for lengths of parametric curves in Section 6.3.

Let $$x = f(t), \qquad y = g(t), \qquad a \le t \le b$$

be a parametric curve in the first quadrant such that the derivatives are continuous and the curve does not retrace its path (Figure 6.4.10).

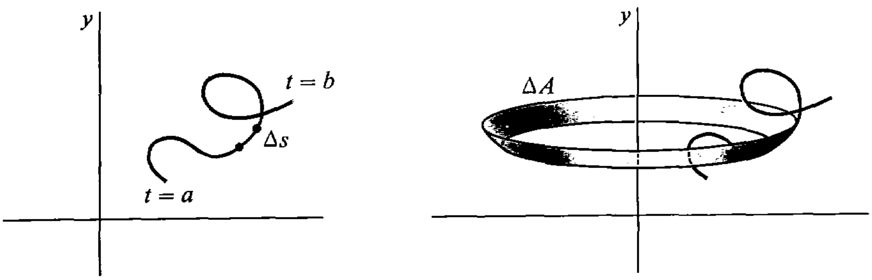


Figure 6.4.10

AREA OF SURFACE OF REVOLUTION

$$A = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (\textit{rotating about } y\textit{-axis}).$$

To justify this new formula we observe that an infinitesimal piece of the surface is almost a cone frustum of radii x, $x + \Delta x$ and slant height Δs. Thus compared to Δt,

$$\Delta s \approx \sqrt{(dx/dt)^2 + (dy/dt)^2}\, \Delta t,$$
$$\Delta A \approx \pi(x + (x + \Delta x))\, \Delta s \approx 2\pi x\, \Delta s,$$
$$\Delta A \approx 2\pi x \sqrt{(dx/dt)^2 + (dy/dt)^2}\, \Delta t.$$

The Infinite Sum Theorem gives the desired formula for area.

This new formula reduces to our first formula when the curve has the simple form $y = f(x)$. If $y = f(x)$, $a \le x \le b$, take $x = t$ and get

$$A = \int_a^b 2\pi x \sqrt{1 + (dy/dx)^2}\, dx \qquad (\text{about } y\text{-axis}).$$

Similarly, if $x = g(y)$, $a \le y \le b$, we take $y = t$ and get the formula

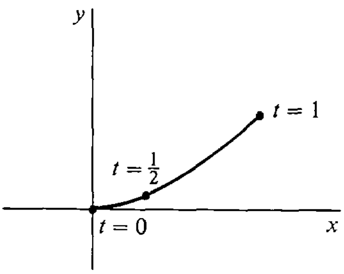
$$A = \int_a^b 2\pi x \sqrt{(dx/dy)^2 + 1}\, dy \qquad (\text{about } y\text{-axis}).$$

EXAMPLE 3 The curve $x = 2t^2$, $y = t^3$, $0 \le t \le 1$ is rotated about the y-axis. Find the area of the surface of revolution (Figure 6.4.11).

We first find dx/dt and dy/dt and then apply the formula for area.

$$\frac{dx}{dt} = 4t, \qquad \frac{dy}{dt} = 3t^2,$$

$$A = \int_0^1 2\pi x \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$
$$= \int_0^1 4\pi t^2 \sqrt{(4t)^2 + (3t^2)^2}\, dt$$
$$= 4\pi \int_0^1 t^2 \sqrt{16t^2 + 9t^4}\, dt$$
$$= 4\pi \int_0^1 t^3 \sqrt{16 + 9t^2}\, dt.$$



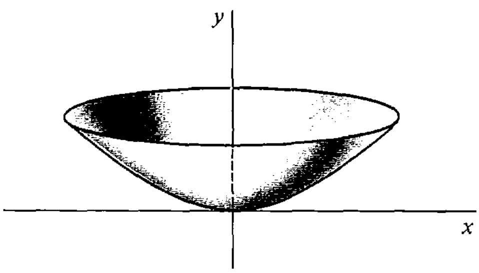


Figure 6.4.11

Let $u = 16 + 9t^2$, $du = 18t\,dt$, $dt = \frac{1}{18t}du$, $t^2 = \frac{u-16}{9}$. Then $u = 16$ at $t = 0$ and $u = 25$ at $t = 1$, so

$$A = 4\pi\int_{16}^{25} t^3\sqrt{u}\frac{1}{18t}\,du = 4\pi\int_{16}^{25}\frac{1}{18}\left(\frac{u-16}{9}\right)\sqrt{u}\,du$$

$$= \frac{2\pi}{81}\int_{16}^{25}(u^{3/2} - 16\sqrt{u})\,du = \frac{5692}{1215}\pi \sim 4.7\pi.$$

EXAMPLE 4 Derive the formula $A = 4\pi r^2$ for the area of the surface of a sphere of radius r.

When the portion of the circle $x^2 + y^2 = r^2$ in the first quadrant is rotated about the y-axis it will form a hemisphere of radius r (Figure 6.4.12). The surface of the sphere has twice the area of this hemisphere.

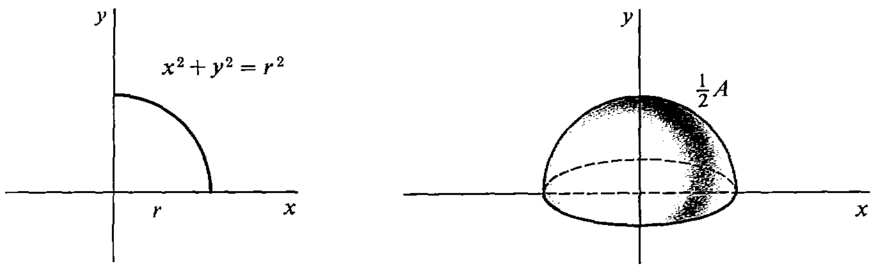


Figure 6.4.12

It is simpler to take y as the independent variable, so the curve has the equation

$$x = \sqrt{r^2 - y^2}, \qquad 0 \le y \le r.$$

Then
$$\frac{dx}{dy} = -\frac{y}{\sqrt{r^2 - y^2}}.$$

This derivative is undefined at $y = 0$. To get around this difficulty we let $0 < a < r$ and divide the surface into the two parts shown in Figure 6.4.13, the surface B generated by the curve from $y = 0$ to $y = a$ and the surface C generated by the curve from $y = a$ to $y = r$.

The area of C is

$$C = \int_a^r 2\pi x\sqrt{(dx/dy)^2 + 1}\,dy$$

$$= \int_a^r 2\pi\sqrt{r^2 - y^2}\sqrt{1 + y^2/(r^2 - y^2)}\,dy$$

$$= \int_a^r 2\pi\sqrt{r^2 - y^2}\sqrt{r^2/(r^2 - y^2)}\,dy$$

$$= \int_a^r 2\pi r\,dy = 2\pi r y\Big]_a^r = 2\pi r(r - a).$$

We could find the area of B by taking x as the independent variable. However,

Figure 6.4.13

it is simpler to let a be an infinitesimal ε. Then B is an infinitely thin ring-shaped surface, so its area is infinitesimal. Therefore the hemisphere has area

$$\tfrac{1}{2}A = B + C \approx 0 + 2\pi r(r - \varepsilon) \approx 2\pi r^2,$$

so $$\tfrac{1}{2}A = 2\pi r^2,$$

and the sphere has area $A = 4\pi r^2$.

If a curve is rotated about the x-axis instead of the y-axis (Figure 6.4.14), we interchange x and y in the formulas for surface area,

$$A = \int_a^b 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt \qquad \text{(about } x\text{-axis)},$$

$$A = \int_a^b 2\pi y\sqrt{(dx/dy)^2 + 1}\,dy \qquad \text{(about } x\text{-axis)},$$

$$A = \int_a^b 2\pi y\sqrt{1 + (dy/dx)^2}\,dx \qquad \text{(about } x\text{-axis)}.$$

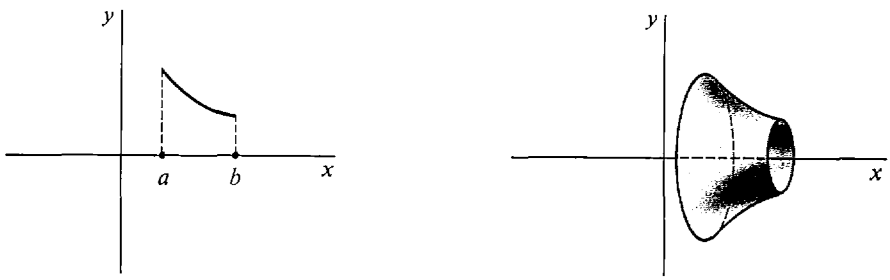


Figure 6.4.14

Most of the time the formula for surface area will give an integral which cannot be evaluated exactly but can only be approximated, for example by the Trapezoidal Rule.

EXAMPLE 5 Let C be the curve

$$y = x^4, \qquad 0 \le x \le 1. \qquad \text{(see Figure 6.4.15)}$$

Set up an integral for the surface area generated by rotating the curve C about (a) the y-axis, (b) the x-axis (see Figure 6.4.16).

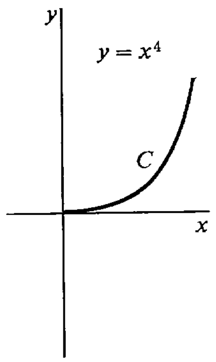
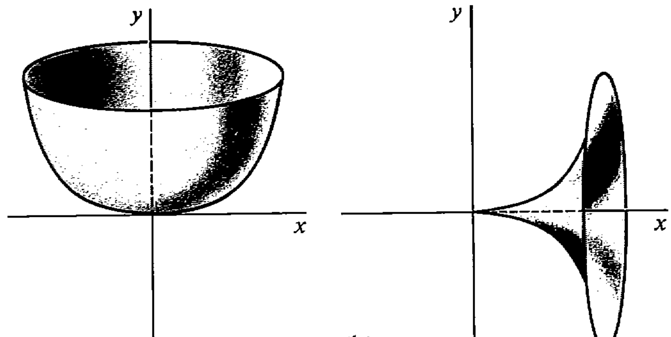


Figure 6.4.15



(a) (b)

Figure 6.4.16

(a)
$$dy/dx = 4x^3$$
$$A = \int_0^1 2\pi x\sqrt{1 + (dy/dx)^2}\, dx$$
$$= \int_0^1 2\pi x\sqrt{1 + 16x^6}\, dx.$$

We cannot evaluate this integral, so we leave it in the above form. The Trapezoidal Rule can be used to get approximate values. When $\Delta x = \frac{1}{10}$ the Trapezoidal Approximation is

$$A \sim 6.42, \text{ error} \le 0.26.$$

(b)
$$A = \int_0^1 2\pi y\sqrt{1 + (dy/dx)^2}\, dx$$
$$= \int_0^1 2\pi x^4\sqrt{1 + 16x^6}\, dx.$$

The Trapezoidal Approximation when $\Delta x = \frac{1}{10}$ is

$$A \sim 3.582 \qquad \text{error} \le 0.9.$$

PROBLEMS FOR SECTION 6.4

In Problems 1–12, find the area of the surface generated by rotating the given curve about the y-axis.

1 $y = x^2, \quad 0 \le x \le 2$

2 $y = cx + d, \quad a \le x \le b$

3 $y = 2x^{3/2}, \quad 0 \le x \le 1$

4 $y = \frac{1}{3}(x^2 + 2)^{3/2}, \quad 1 \le x \le 2$

5 $y = \frac{1}{3}x\sqrt{x} - \sqrt{x}, \quad 1 \le x \le 4$

6 $y = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}, \quad 1 \le x \le 2$

7 $y = \frac{3}{5}x^{5/3} - \frac{3}{4}x^{1/3}, \quad 1 \le x \le 8$

8 $x = 2t + 1, y = 4 - t, \quad 0 \le t \le 4$

9 $x = t + 1, y = \frac{1}{2}t^2 + t, \quad 0 \le t \le 2$

10 $x = t^2, y = \frac{1}{3}t^3, \quad 0 \le t \le 3$

11 $x = t^3, y = 3t + 1, \quad 0 \le t \le 1$

12 $x^{2/3} + y^{2/3} = 1$, first quadrant

In Problems 13–20, find the area of the surface generated by rotating the given curve about the x-axis.

13 $y = \frac{1}{3}x^3, \quad 0 \le x \le 1$

14 $y = \sqrt{x}, \quad 1 \le x \le 2$

15 $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \quad 1 \le x \le 2$

16 $y = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}, \quad 1 \le x \le 2$

17 $y = \frac{1}{3}x\sqrt{x} - \sqrt{x}, \quad 3 \le x \le 4$

18 $y = \frac{3}{5}x^{5/3} - \frac{3}{4}x^{1/3}, \quad 8 \le x \le 27$

19 $x = 2t + 1, y = 4 - t, \quad 0 \le t \le 4$

20 $x = t^2 + t, y = 2t + 1, \quad 0 \le t \le 1$

21 The part of the circle $x^2 + y^2 = r^2$ between $x = 0$ and $x = a$ in the first quadrant is rotated about the x-axis. Find the area of the resulting zone of the sphere $(0 < a < r)$.

22 Solve the above problem when the rotation is about the y-axis.

In Problems 23–26 set up integrals for the areas generated by rotating the given curve about (a) the y-axis, (b) the x-axis.

23 $y = x^5, \quad 0 \le x \le 1$

24 $x = y + \sqrt{y}, \quad 2 \le y \le 3$

25 $x = t^2 + t, y = t^2 - 1, \quad 1 \le t \le 10$

26 $x = t^4, y = t^3, \quad 2 \le t \le 4$

27 Set up an integral for the area generated by rotating the curve $y = \frac{1}{2}x^2, 0 \le x \le 1$ about the x-axis and find the Trapezoidal Approximation with $\Delta x = 0.2$.

28 Set up an integral for the area generated by rotating the curve $y = \frac{1}{3}x^3, 0 \le x \le 1$ about the y-axis and find the Trapezoidal Approximation with $\Delta x = 0.2$.

□ 29 Show that the surface area of the torus generated by rotating the circle of radius r and center $(c, 0)$ about the y-axis $(r < c)$ is $A = 4\pi^2 rc$. *Hint:* Take y as the independent variable and use the formula $\int_a^b r\,dy/\sqrt{r^2 - y^2}$ for the length of the arc of the circle from $y = a$ to $y = b$.

6.5 AVERAGES

Given n numbers $y_1, \ldots, y_n$, their average value is defined as

$$y_{\text{ave}} = \frac{y_1 + \cdots + y_n}{n}.$$

If all the y_i are replaced by the average value y_{ave}, the sum will be unchanged,

$$y_1 + \cdots + y_n = y_{\text{ave}} + \cdots + y_{\text{ave}} = ny_{\text{ave}}.$$

If f is a continuous function on a closed interval $[a, b]$, what is meant by the average value of f between a and b (Figure 6.5.1)? Let us try to imitate the procedure for finding the average of n numbers. Take an infinite hyperreal number H and divide the interval $[a, b]$ into infinitesimal subintervals of length $dx = (b - a)/H$. Let

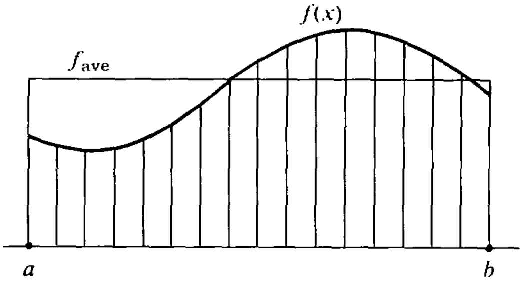


Figure 6.5.1

us "sample" the value of f at the H points $a, a + dx, a + 2\,dx, \ldots, a + (H - 1)\,dx$. Then the average value of f should be infinitely close to the sum of the values of f at $a, a + dx, \ldots, a + (H - 1)\,dx$, divided by H. Thus

$$f_{ave} \approx \frac{f(a) + f(a + dx) + f(a + 2\,dx) + \cdots + f(a + (H - 1)\,dx)}{H}.$$

Since $dx = \dfrac{b - a}{H}$, $\dfrac{1}{H} = \dfrac{dx}{b - a}$ and we have

$$f_{ave} \approx \frac{f(a)\,dx + f(a + dx)\,dx + \cdots + f(a + (H - 1)\,dx)\,dx}{b - a},$$

$$f_{ave} \approx \frac{\sum_a^b f(x)\,dx}{b - a}.$$

Taking standard parts, we are led to

DEFINITION

*Let f be continuous on $[a, b]$. The **average value** of f between a and b is*

$$f_{ave} = \frac{\int_a^b f(x)\,dx}{b - a}.$$

Geometrically, the area under the curve $y = f(x)$ is equal to the area under the constant curve $y = f_{ave}$ between a and b,

$$f_{ave} \cdot (b - a) = \int_a^b f(x)\,dx.$$

EXAMPLE 1 Find the average value of $y = \sqrt{x}$ from $x = 1$ to $x = 4$ (Figure 6.5.2).

$$y_{ave} = \frac{\int_1^4 \sqrt{x}\,dx}{(4 - 1)} = \frac{\frac{2}{3}x^{3/2}]_1^4}{3} = \frac{\frac{2}{3}(8 - 1)}{3} = \frac{14}{9}.$$

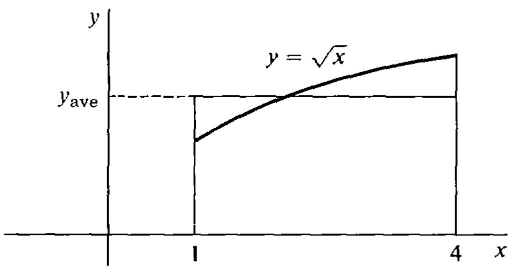


Figure 6.5.2

Recall that in Section 3.8, we defined the *average slope* of a function F between a and b as the quotient

$$\text{average slope} = \frac{F(b) - F(a)}{b - a}.$$

Using the Fundamental Theorem of Calculus we can find the connection between the average value of F' and the average slope of F.

THEOREM 1

Let F be an antiderivative of a continuous function f on an open interval I. Then for any $a < b$ in I, the average slope of F between a and b is equal to the average value of f between a and b,

$$\frac{F(b) - F(a)}{b - a} = \frac{\int_a^b f(x)\,dx}{b - a}.$$

PROOF By the Fundamental Theorem,

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

THEOREM 2 (Mean Value Theorem for Integrals)

Let f be continuous on $[a, b]$. Then there is a point c strictly between a and b where the value of f is equal to its average value,

$$f(c) = \frac{\int_a^b f(x)\,dx}{b - a}.$$

PROOF Theorem 2 is illustrated in Figure 6.5.3. We can make f continuous on the whole real line by defining $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. By the Second Fundamental Theorem of Calculus, f has an antiderivative F. By the Mean Value Theorem there is a point c strictly between a and b at which $F'(c)$ is equal to the average slope of F,

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

But $F'(c) = f(c)$ and $F(b) - F(a) = \int_a^b f(x)\,dx$, so

$$f(c) = \frac{\int_a^b f(x)\,dx}{b - a}.$$

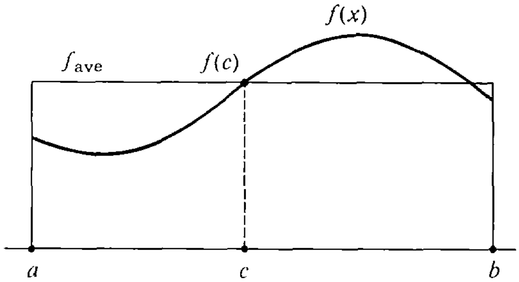


Figure 6.5.3

EXAMPLE 2 A car starts at rest and moves with velocity $v = 3t^2$. Find its average velocity between times $t = 0$ and $t = 5$. At what point of time is its velocity equal to the average velocity?

$$v_{\text{ave}} = \frac{\int_0^5 3t^2\,dt}{5 - 0} = \frac{t^3]_0^5}{5} = \frac{125}{5} = 25.$$

To find the value of t where $v = v_{\text{ave}}$, we put

$$3t^2 = 25, \qquad t = \sqrt{25/3} = 5/\sqrt{3}.$$

Suppose a car drives from city A to city B and back, a distance of 120 miles each way. From A to B it travels at a speed of 30 mph, and on the return trip it travels at 60 mph. What is the average speed?

If we choose distance as the independent variable we get one answer, and if we choose time we get another.

Average speed with respect to time: The car takes $120/30 = 4$ hours to go from A to B and $120/60 = 2$ hours to return to A. The total trip takes 6 hours.

$$v_{\text{ave}} = \frac{30 \cdot 4 + 60 \cdot 2}{6} = \frac{240}{6} = 40 \text{ mph}.$$

Average speed with respect to distance: The car goes 120 miles at 30 mph and 120 miles at 60 mph, with a total distance of 240 miles. Therefore

$$v_{\text{ave}} = \frac{30 \cdot 120 + 60 \cdot 120}{240} = 45 \text{ mph}.$$

From Figure 6.5.4 we see that the average with respect to time is smaller because most of the time was spent at the lower speed of 30 mph.

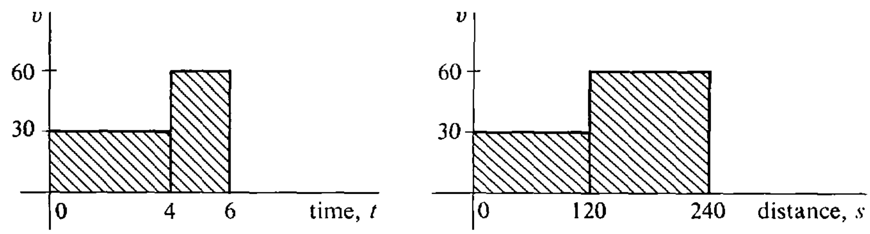
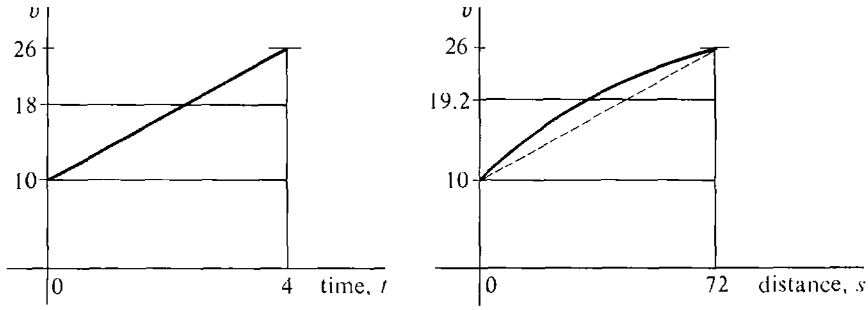


Figure 6.5.4

In general, if y is given both as a function of s and of t, $y = f(s) = g(t)$, then there is one average of y with respect to s, and another with respect to t.

EXAMPLE 3 A car travels with velocity $v = 4t + 10$, where t is time. Between times $t = 0$ and $t = 4$ find the average velocity with respect to (a) time, and (b) distance.

(a) $$v_{\text{ave}} = \frac{\int_0^4 4t + 10\,dt}{4} = \frac{2t^2 + 10t]_0^4}{4} = 18 \quad \text{(Figure 6.5.5(a))}.$$



(a) (b)

Figure 6.5.5

(b) Let s be the distance, and put $s = 0$ when $t = 0$. Since $ds = v\,dt = (4t + 10)\,dt$, at time $t = 4$ we have

$$s = \int_0^4 4t + 10\,dt = 2t^2 + 10t\Big]_0^4 = 72.$$

Then
$$v_{\text{ave}} = \frac{\int_0^{72}(4t+10)\,ds}{72} = \frac{\int_0^4 (4t+10)(4t+10)\,dt}{72}$$
$$= \frac{\int_0^4 16t^2 + 80t + 100\,dt}{72} = \frac{\frac{16}{3}t^3 + 40t^2 + 100t]_0^4}{72}$$
$$= \frac{1024/3 + 640 + 400}{72} \sim 19.2 \text{ (Figure 6.5.5(b))}.$$

PROBLEMS FOR SECTION 6.5

In Problems 1–8, sketch the curve, find the average value of the function, and sketch the rectangle which has the same area as the region under the curve.

1	$f(x) = 1 + x, \quad -1 \le x \le 1$	2	$f(x) = 2 - \frac{1}{2}x, \quad 0 \le x \le 4$
3	$f(x) = 4 - x^2, \quad -2 \le x \le 2$	4	$f(x) = 1 + x^2, \quad -2 \le x \le 2$
5	$f(x) = \sqrt{2x - 1}, \quad 1 \le x \le 5$	6	$f(x) = x^3, \quad 0 \le x \le 2$
7	$f(x) = \sqrt[3]{x}, \quad 0 \le x \le 8$	8	$f(x) = 1 - x^4, \quad -1 \le x \le 1$

In Problems 9–22, find the average value of $f(x)$.

9	$f(x) = x^2 - \sqrt{x}, \quad 0 \le x \le 3$	10	$f(x) = \sqrt{x} + 1/\sqrt{x}, \quad 1 \le x \le 9$
11	$f(x) = 6x, \quad -4 \le x \le 2$	12	$f(x) = \dfrac{3x}{\sqrt{1 - x^2}}, \quad -\frac{1}{2} \le x \le \frac{1}{2}$
13	$f(x) = 2x\sqrt{1 + x^2}, \quad -3 \le x \le 3$	14	$f(x) = 5x^4 - 8x^3 + 10, \quad 0 \le x \le 10$
15	$f(x) = \sin x, \quad 0 \le x \le \pi$	16	$f(x) = \sin x, \quad 0 \le x \le 2\pi$
17	$f(x) = \sin x \cos x, \quad 0 \le x \le \pi/2$	18	$f(x) = x + \sin x, \quad 0 \le x \le 2\pi$
19	$f(x) = e^x, \quad -1 \le x \le 1$	20	$f(x) = e^x - 2x, \quad 0 \le x \le 2$
21	$f(x) = \dfrac{1}{x}, \quad 1 \le x \le 4$	22	$f(x) = \dfrac{x}{x + 1}, \quad 0 \le x \le 4$

In Problems 23–28, find a point c in the given interval such that $f(c)$ is equal to the average value of $f(x)$.

23	$f(x) = 2x, \quad -4 \le x \le 6$	**24**	$f(x) = 3x^2, \quad 0 \le x \le 3$
25	$f(x) = \sqrt{2x}, \quad 0 \le x \le 2$	**26**	$f(x) = x^2 - x, \quad -1 \le x \le 1$
27	$f(x) = x^{2/3}, \quad 0 \le x \le 2$	**28**	$f(x) = \lvert x - 3 \rvert, \quad 1 \le x \le 4$

29 What is the average distance between a point x in the interval $[5, 8]$ and the origin?

30 What is the average distance between a point in the interval $[-4, 3]$ and the origin?

31 Find the average distance from the origin to a point on the curve $y = x^{3/2}$, $0 \le x \le 3$, with respect to x.

32 A particle moves with velocity $v = 6t$ from time $t = 0$ to $t = 10$. Find its average velocity with respect to (a) time, (b) distance.

33 An object moves with velocity $v = t^3$ from time $t = 0$ to $t = 2$. Find its average velocity with respect to (a) time, (b) distance.

□ **34** A particle moves with positive velocity $v = f(t)$ from $t = a$ to $t = b$. Thus its average velocity with respect to time is

$$\frac{\int_a^b f(t)\,dt}{(b - a)}.$$

Show that its average velocity with respect to distance is

$$\frac{\int_a^b (f(t))^2\,dt}{\int_a^b f(t)\,dt}.$$

6.6 SOME APPLICATIONS TO PHYSICS

The Infinite Sum Theorem can frequently be used to derive formulas in physics.

1 MASS AND DENSITY, ONE DIMENSION

Consider a one-dimensional object such as a length of wire. We ignore the atomic nature of matter and assume that it is distributed continuously along a line segment. If the density ρ per unit length is the same at each point of the wire, then the mass is the product of the density and the length, $m = \rho L$. If L is in centimeters and ρ in grams per centimeter, then m is in grams. (ρ is the Greek letter "rho".)

Now suppose that the density of the wire varies continuously with the position. Put the wire on the x-axis between the points $x = a$ and $x = b$, and let the density at the point x be $\rho(x)$. Consider the piece of the wire of infinitesimal length Δx and mass Δm shown in Figure 6.6.1. At each point between x and $x + \Delta x$, the density is infinitely close to $\rho(x)$, so

$$\Delta m \approx \rho(x)\,\Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

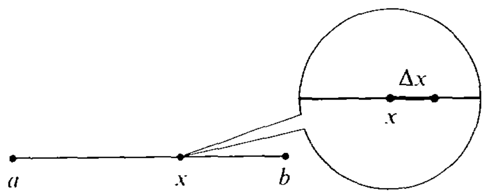


Figure 6.6.1

Therefore by the Infinite Sum Theorem, the total mass is

$$m = \int_a^b \rho(x)\, dx.$$

EXAMPLE 1 Find the mass of a wire 6 cm long whose density at distance x from the center is $9 - x^2$ gm/cm. In Figure 6.6.2, we put the center of the wire at the origin. Then

$$m = \int_{-3}^{3} 9 - x^2\, dx = 9x - \tfrac{1}{3}x^3 \Big]_{-3}^{3} = 36 \text{ gm}.$$

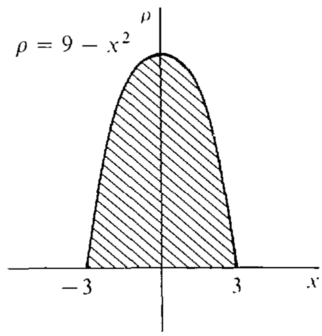


Figure 6.6.2

2 MASS AND DENSITY, TWO DIMENSIONS

Imagine a flat plate which occupies the region below the curve $y = f(x), f(x) \geq 0$, from $x = a$ to $x = b$. If its density per unit area is a constant ρ gm/cm^2, then its mass is the product of the density and area,

$$m = \rho A = \rho \int_a^b f(x)\, dx.$$

Suppose instead that the density depends on the value of x, $\rho(x)$. Consider a vertical strip of the plate of infinitesimal width Δx (Figure 6.6.3). On the strip between x and $x + \Delta x$, the density is everywhere infinitely close to $\rho(x)$, so

$$\Delta m \approx \rho(x)\, \Delta A \approx \rho(x) f(x)\, \Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

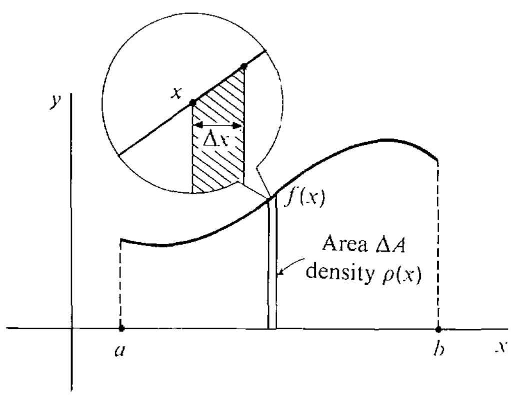


Figure 6.6.3

By the Infinite Sum Theorem,

$$m = \int_a^b \rho(x) f(x)\, dx.$$

EXAMPLE 2 A circular disc of radius r has density at each point equal to the distance of the point from the y-axis. Find its mass. (The center of the circle, shown in Figure 6.6.4, is at the origin.) The circle is the region between the curves $-\sqrt{r^2 - x^2}$ and $\sqrt{r^2 - x^2}$ from $-r$ to r. The density at a point (x, y) in the disc is $|x|$. By symmetry, all four quadrants have the same mass. We shall find the mass m_1 of the first quadrant and multiply by four.

$$m_1 = \int_0^r \sqrt{r^2 - x^2}\, x\, dx.$$

Put $u = r^2 - x^2$, $du = -2x\, dx$; $u = r^2$ when $x = 0$, and $u = 0$ when $x = r$.

$$m_1 = \int_{r^2}^0 -\tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2}\int_0^{r^2} \sqrt{u}\, du = \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_0^{r^2} = \tfrac{1}{3}r^3.$$

Then $m = 4m_1 = \frac{4}{3}r^3$.

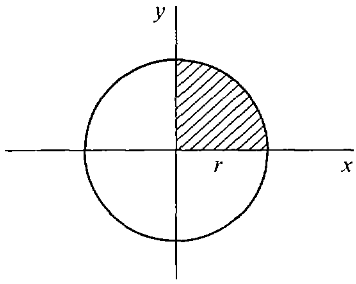


Figure 6.6.4

3 MOMENTS, ONE DIMENSION

Two children on a weightless seesaw will balance perfectly if the product of their masses and their distances from the fulcrum are equal, $m_1 d_1 = m_2 d_2$ (Figure 6.6.5).

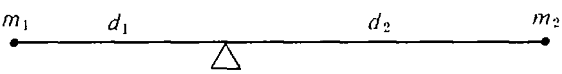


Figure 6.6.5

For example, a 60 lb child 6 feet from the fulcrum will balance a 40 lb child 9 feet from the fulcrum, $60 \cdot 6 = 40 \cdot 9$. If the fulcrum is at the origin $x = 0$, the masses m_1 and m_2 have coordinates $x_1 = -d_1$ and $x_2 = d_2$. The equation for balancing becomes

$$m_1 x_1 + m_2 x_2 = 0.$$

Similarly, finitely many masses $m_1, \ldots, m_k$ at the points $x_1, \ldots, x_k$ will balance about the point $x = 0$ if

$$m_1 x_1 + \cdots + m_k x_k = 0.$$

Given a mass m at the point x, the quantity mx is called the *moment about the origin.*

The moment of a finite collection of point masses $m_1, \ldots, m_k$ at $x_1, \ldots, x_k$ about the origin is defined as the sum

$$M = m_1 x_1 + \cdots + m_k x_k.$$

Suppose the point masses are rigidly connected to a rod of mass zero. If the moment M is equal to zero, the masses will balance at the origin. In general they will balance at a point $\bar{x}$ called the *center of gravity* (Figure 6.6.6). $\bar{x}$ is equal to the moment divided by the total mass m,

$$\bar{x} = \frac{M}{m} = \frac{m_1 x_1 + \cdots + m_k x_k}{m_1 + \cdots + m_k}.$$

Since the mass m is positive, the moment M has the same sign as the center of gravity $\bar{x}$.

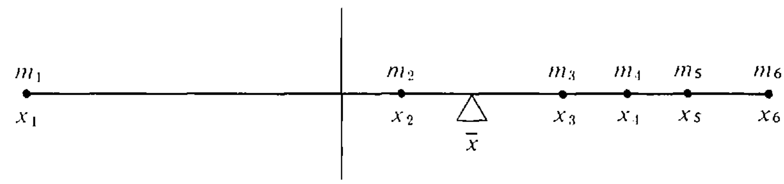


Figure 6.6.6

Now consider a length of wire between $x = a$ and $x = b$ whose density at x is $\rho(x)$. The *moment* of the wire about the origin is defined as the integral

$$M = \int_a^b x\rho(x)\,dx.$$

This formula is justified by considering a piece of the wire of infinitesimal length Δx. On the piece from x to $x + \Delta x$ the density remains infinitely close to $\rho(x)$. Thus if ΔM is the moment of the piece,

$$\Delta M \approx x\,\Delta m \approx x\rho(x)\,\Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

The moment of an object is equal to the sum of the moments of its parts. Hence by the Infinite Sum Theorem,

$$M = \int_a^b x\rho(x)\,dx.$$

If the wire has moment M about the origin and mass m, the *center of mass* of the wire is defined as the point

$$\bar{x} = M/m.$$

A point of mass m located at $\bar{x}$ has the same moment about the origin as the whole wire, $M = \bar{x}m$. Physically, the wire will balance on a fulcrum placed at the center of mass.

EXAMPLE 3 A wire between $x = 0$ and $x = 1$ has density $\rho(x) = x^2$ (Figure 6.6.7). The moment is

$$M = \int_0^1 x^2 x\,dx = \frac{x^4}{4}\bigg]_0^1 = \frac{1}{4}.$$

The mass and center of mass are

$$m = \int_0^1 x^2\,dx = \tfrac{1}{3}, \qquad \bar{x} = M/m = \tfrac{3}{4}.$$

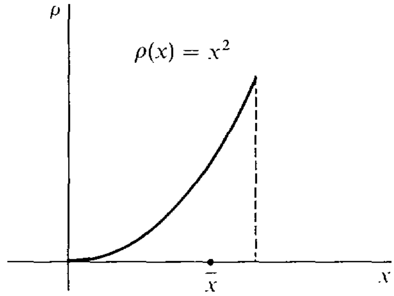


Figure 6.6.7

4 MOMENTS, TWO DIMENSIONS

A mass m at the point (x_0, y_0) in the (x, y) plane will have moments M_x about the x-axis and M_y about the y-axis (Figure 6.6.8). They are defined by

$$M_x = my_0, \qquad M_y = mx_0.$$

Consider a vertical length of wire of mass m and constant density which lies on the line $x = x_0$ from $y = a$ to $y = b$.

The wire has density

$$\rho = \frac{m}{b - a}.$$

The infinitesimal piece of the wire from y to $y + \Delta y$ shown in Figure 6.6.9 will have mass and moments

$$\begin{aligned} \Delta m &= \rho\,\Delta y, \\ \Delta M_x &\approx y\,\Delta m = y\rho\,\Delta y \quad \text{(compared to } \Delta y), \\ \Delta M_y &\approx x_0\,\Delta m = x_0\rho\,\Delta y \quad \text{(compared to } \Delta y). \end{aligned}$$

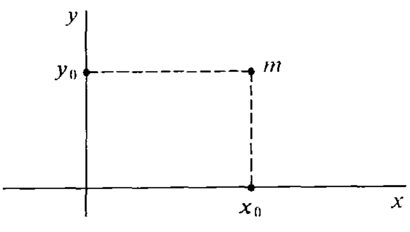


Figure 6.6.8

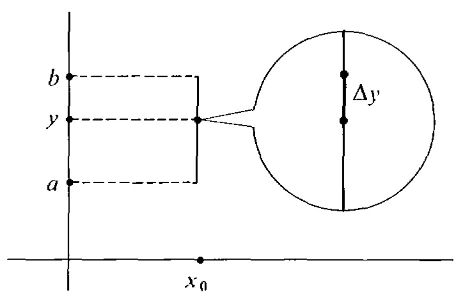


Figure 6.6.9

The Infinite Sum Theorem gives the moments for the whole wire,

$$M_x = \int_a^b y\rho\, dy = \rho\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right) = \frac{1}{2}(b + a)m,$$

$$M_y = \int_a^b x_0\rho\, dy = x_0\rho(b - a) = x_0 m.$$

We next take up the case of a flat plate which occupies the region R under the curve $y = f(x), f(x) \geq 0$, from $x = a$ to $x = b$ (Figure 6.6.10). Assume the density

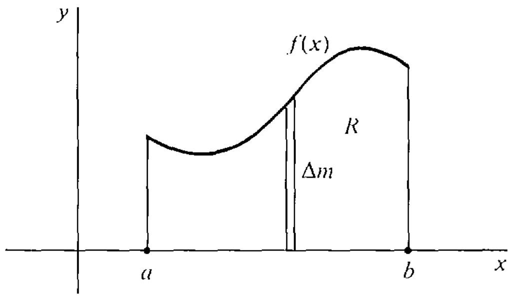


Figure 6.6.10

$\rho(x)$ depends only on the x-coordinate. A vertical slice of infinitesimal width Δx between x and $x + \Delta x$ is almost a vertical length of wire between 0 and $f(x)$ which has area ΔA and mass $\Delta m \approx \rho(x)\,\Delta A \approx \rho(x)f(x)\,\Delta x$ (compared to Δx). Putting the mass Δm into the vertical wire formulas, the moments are

$$\Delta M_y \approx x\,\Delta m \approx x\rho(x)f(x)\,\Delta x \qquad \text{(compared to } \Delta x\text{)},$$

$$\Delta M_x \approx \tfrac{1}{2}(f(x) + 0)\,\Delta m \approx \tfrac{1}{2}\rho(x)f(x)^2\,\Delta x \qquad \text{(compared to } \Delta x\text{)}.$$

Then by the Infinite Sum Theorem, the total moments are

$$M_y = \int_a^b x\rho(x)f(x)\,dx,$$

$$M_x = \int_a^b \tfrac{1}{2}\rho(x)f(x)^2\,dx.$$

The *center of mass* of a two-dimensional object is defined as the point $(\bar{x}, \bar{y})$ with coordinates

$$\bar{x} = M_y/m, \qquad \bar{y} = M_x/m.$$

A single mass m at the point $(\bar{x}, \bar{y})$ will have the same moments as the two-dimensional body, $M_x = m\bar{y}$, $M_y = m\bar{x}$. The object will balance on a pin placed at the center of mass.

If a two-dimensional object has constant density, the center of mass depends only on the region R which it occupies. The *centroid* of a region R is defined as the center of mass of an object of constant density which occupies R. Thus if R is the region below the continuous curve $y = f(x)$ from $x = a$ to $x = b$, then the centroid has coordinates

$$\bar{x} = \int_a^b xf(x)\,dx/A, \qquad \bar{y} = \int_a^b \tfrac{1}{2}f(x)^2\,dx/A,$$

where A is the area $A = \int_a^b f(x)\,dx$.

EXAMPLE 4 Find the centroid of the triangular region R bounded by the x-axis, the y-axis, and the line $y = 1 - \frac{1}{2}x$ shown in Figure 6.6.11. R is the region under the curve $y = 1 - \frac{1}{2}x$ from $x = 0$ to $x = 2$. The area of R is

$$A = \int_0^2 1 - \tfrac{1}{2}x\,dx = x - \tfrac{1}{4}x^2\Big]_0^2 = 1.$$

The centroid is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \int_0^2 x(1 - \tfrac{1}{2}x)\,dx = \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3\Big]_0^2 = \tfrac{2}{3},$$

$$\bar{y} = \int_0^2 \tfrac{1}{2}(1 - \tfrac{1}{2}x)^2\,dx = \int_0^2 \tfrac{1}{2} - \tfrac{1}{2}x + \tfrac{1}{8}x^2\,dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{24}x^3\Big]_0^2 = \tfrac{1}{3}.$$

Thus the centroid is the point $(\frac{2}{3}, \frac{1}{3})$.

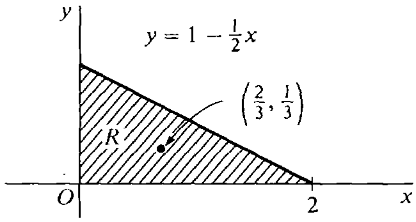


Figure 6.6.11

The following principle often simplifies a problem in moments.

If an object is symmetrical about an axis, then its moment about that axis is zero and its center of mass lies on the axis.

PROOF Consider the y-axis. Suppose a plane object occupies the region under the curve $y = f(y)$ from $-a$ to a and its density at a point (x, y) is $\rho(x)$ (Figure 6.6.12). The object is symmetric about the y-axis, so for all x between 0 and a,

$$f(-x) = f(x), \qquad \rho(-x) = \rho(x).$$

Then $$M_y = \int_{-a}^{a} xf(x)\rho(x)\,dx = \int_{-a}^{0} xf(x)\rho(x)\,dx + \int_0^a xf(x)\rho(x)\,dx$$

$$= \int_a^0 (-x)f(-x)\rho(-x)\,d(-x) + \int_0^a xf(x)\rho(x)\,dx = 0.$$

Also, $\bar{x} = M_y/m = 0$.

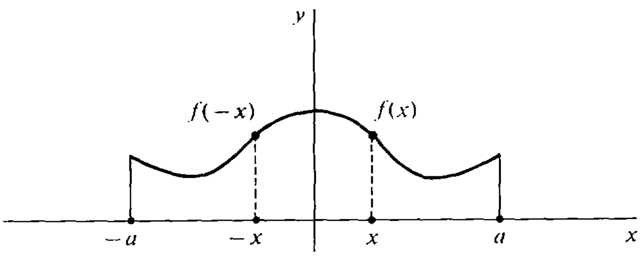


Figure 6.6.12 Symmetry about the y-axis

EXAMPLE 5 Find the centroid of the semicircle $y = \sqrt{1 - x^2}$ (Figure 6.6.13). By symmetry, the centroid is on the y-axis, $\bar{x} = 0$. The area of the semicircle is $\frac{1}{2}\pi$. Then

$$\frac{1}{2}\pi\bar{y} = \int_{-1}^{1} \frac{1}{2}(1 - x^2)\,dx = \frac{1}{2}x - \frac{1}{6}x^3\Big]_{-1}^{1} = \frac{2}{3}, \qquad \bar{y} = \frac{2/3}{\frac{1}{2}\pi} = \frac{4}{3\pi}.$$

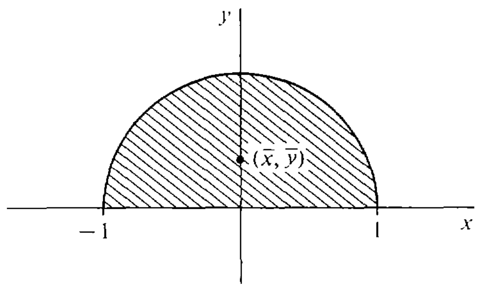


Figure 6.6.13

5 WORK

A constant force F acting along a straight line for a distance s requires the amount of work

$$W = Fs.$$

For example, the force of gravity on an object of mass m near the surface of the earth is very nearly a constant g times the mass, $F = gm$. Thus to lift an object of mass m a distance s against gravity requires the work $W = gms$. The following principle is useful in computing work done against gravity.

The amount of work done against gravity to move an object is the same as it would be if all the mass were concentrated at the center of mass. Moreover, the work against gravity depends only on the vertical change in position of the center of mass, not on the actual path of its motion.

That is, $W = gms$ where s is the vertical change in the center of mass.

EXAMPLE 6 A semicircular plate of radius one, constant density, and mass m lies flat on the table. (a) How much work is required to stand it up with the straight edge horizontal on the table (Figure 6.6.14(a))? (b) How much work is required to stand it up with the straight edge vertical and one corner on the table (Figure 6.6.14(b))? From the previous exercise, we know that the

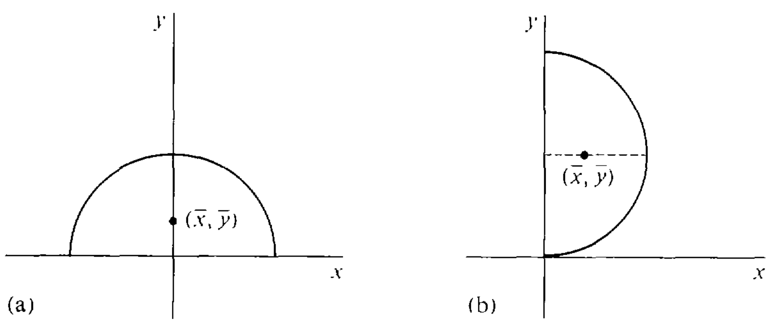


Figure 6.6.14

center of mass is on the central radius $4/3\pi$ from the center of the circle. Put the x-axis on the surface of the table.

(a) The center of mass is lifted a distance $4/3\pi$ above the table. Therefore $W = mg \cdot 4/(3\pi)$.

(b) The center of mass is lifted a distance 1 above the table, so $W = mg$.

Suppose a force $F(s)$ varies continuously with the position s and acts on an object to move it from $s = a$ to $s = b$. The work is then the definite integral of the force with respect to s,

$$W = \int_a^b F(s)\,ds.$$

To justify this formula we consider an infinitesimal length Δs. On the interval from s to $s + \Delta s$ the force is infinitely close to $F(s)$, so the work ΔW done on this interval satisfies

$$\Delta W \approx F(s)\,\Delta s \qquad \text{(compared to } \Delta s\text{)}.$$

By the Infinite Sum Theorem,

$$W = \int_a^b F(s)\,ds.$$

EXAMPLE 7 A spring, shown in Figure 6.6.15, of natural length L exerts a force $F = cx$ when compressed a distance x. Find the work done in compressing the spring from length $L - a$ to length $L - b$.

$$W = \int_a^b cx\,dx = \tfrac{1}{2}cx^2\Big]_a^b = \tfrac{1}{2}c(b^2 - a^2).$$

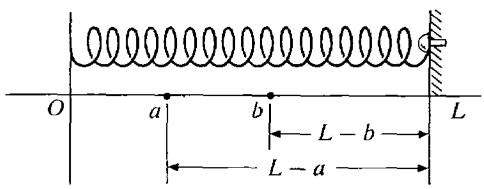


Figure 6.6.15

EXAMPLE 8 The force of gravity between two particles of mass m_1 and m_2 is

$$F = gm_1m_2/s^2,$$

where g is a constant and s is the distance between the particles. Find the work required to move the particle m_2 from a distance a to a distance b from m_1 (Figure 6.6.16).

$$W = \int_a^b F\,ds = \int_a^b \frac{gm_1m_2}{s^2}\,ds = gm_1m_2(-s^{-1})\Big]_a^b = gm_1m_2\left(\frac{1}{a} - \frac{1}{b}\right).$$

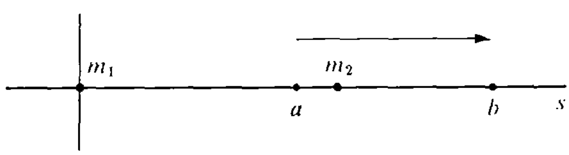


Figure 6.6.16

PROBLEMS FOR SECTION 6.6

In Problems 1–16 below, find (a) the mass, (b) the moments about the x- and y-axes, (c) the center of mass of the given object.

1 A wire on the x-axis, $0 \le x \le 2$ with density $\rho(x) = 2$.

2 A wire on the x-axis, $0 \le x \le 4$, with density $\rho(x) = x^3$.

3 A wire on the y-axis, $0 \le y \le 4$, whose density is twice the distance from the lower end of the wire times the square of the distance from the upper end.

4 A straight wire from the point (0, 0) to the point (1, 1) whose density at each point (x, x) is equal to $3x$.

5 A wire of length 6 and constant density k which is bent in the shape of an L covering the intervals [0, 2] on the x-axis and [0, 4] on the y-axis.

6 The plane object bounded by the x-axis and the curve $y = 4 - x^2$, with constant density k.

7 The plane object bounded by the x-axis and the curve $y = 4 - x^2$, with density $\rho(x) = x^2$.

8 The plane object bounded by the lines $x = 0$, $y = x$, $y = 4 - 3x$, with density $\rho(x) = 2x$.

9 The plane object between the x-axis and the curve $y = x^2$, $0 \le x \le 1$, with density $\rho(x) = 1/x$.

10 The object bounded by the x-axis and the curve $y = x^3$, $0 \le x \le 1$, with density $\rho(x) = 1 - x^2$.

11 The object bounded by the x-axis and the curve $y = 1/x$, $1 \le x \le 2$, with density $\rho(x) = \sqrt{x}$.

12 The disc bounded by $x^2 + y^2 = 4$ with density $\rho(x) = \sqrt{4 - x^2}$.

13 The object in the top half of the circle $x^2 + y^2 = 1$, with density $\rho(x) = 2|x|$.

14 The object between the x-axis and the curve $y = \sqrt{1 - x^4}$, with density equal to the cube of the distance from the y-axis.

15 The object bounded by the x-axis and the curve $y = 4x - x^2$, with density $\rho(x) = 2x$.

16 The object bounded by the curves $y = -f(x)$ and $y = f(x)$, $0 \le x \le 3$, with density $\rho(x) = 4/f(x)$. ($f(x)$ is always positive.)

In Problems 17–24, sketch and find the centroid of the region bounded by the given curves.

17 $y = 0, \quad y = 2, \quad -1 \le x \le 5$

18 $y = 0, \quad x = 0, \quad 3x + 4y = 12$

19 $y = 0, \quad y = 1 - x^2$

20 $y = 0, \quad y = 1 - x^2, \quad 0 \le x \le 1$

21 $y = 0, \quad y = \sqrt{9 - x^2}$

22 $y = 0, \quad y = \sqrt{9 - x^2}, \quad 0 \le x \le 3$

23 $y = 0, \quad y = x^{1/3}, \quad 0 \le x \le 1$

24 $x = 0, \quad y = 0, \quad \sqrt{x} + \sqrt{y} = 1$, first quadrant

25 Find the mass of an object in the region under the curve $y = \sin x$, $0 \le x \le \pi$, with density $\rho(x) = \cos^2 x$.

26 Find the mass of an object in the region between the curves $y = \sin x \cos x$, $y = \sin x$, $0 \le x \le \pi/2$, with density $\rho(x) = \cos x$.

27 Find the mass of an object in the region under the curve $y = e^x$, $-1 \le x \le 1$, with density e^{1-2x}.

28 Find the mass of an object in the region under the curve $y = \ln x$, $1 \le x \le e$, with density $\rho(x) = 1/x$.

29 Find the centroid of the region under the curve $y = x^{-2}$, $1 \le x \le 2$.

30 Find the centroid of the region under the curve $y = 1/\sqrt{x}$, $1 \le x \le 4$.

31 Find the centroid of the region bounded by $y = 0$, $y = x(1 - x^2)$, $0 \le x \le 1$.

□ 32 Show that the moments of an object bounded by the two curves $y = f(x)$ and $y = g(x)$, $a \le x \le b$, are

$$M_x = \int_a^b \tfrac{1}{2}\rho(x)(g(x)^2 - f(x)^2)\,dx, \qquad M_y = \int_a^b x\rho(x)(g(x) - f(x))\,dx.$$

33 Use the formulas in Problem 32 to find the centroid of the region between the curves $y = x^2$ and $y = x$.

34 A piece of metal weighing 50 lbs is in the shape of a triangle of sides 3, 4, and 5 ft. Find the amount of work required to stand the piece up on (a) the 3 ft side, (b) the 4 ft side.

35 A 4 ft chain lies flat on the ground and has constant density of 5 lbs/ft. How much work is required to lift one end 6 ft above the ground?

36 In Problem 35, how much work is required to lift the center of the chain 6 ft above the ground?

37 A 4 ft chain has a density of $4x$ lbs/ft at a point x ft from the left end. How much work is needed to lift the left end 6 ft above the ground?

38 In Problem 37, how much work is needed to lift both ends of the chain to the same point 6 ft above the ground?

39 A spring exerts a force of $4x$ lbs when compressed a distance x. How much work is needed to compress the spring 5 ft from its natural length?

40 A bucket of water weighs 10 lbs and is tied to a rope which has a density of $\frac{1}{10}$ lb/ft. How much work is needed to lift the bucket from the bottom of a 20 ft well?

41 The bucket in Problem 40 is leaking water at the rate of $\frac{1}{10}$ lb/sec and is raised from the well bottom at the rate of 4 ft/sec. How much work is expended in lifting the bucket?

42 Two electrons repel each other with a force inversely proportional to the square of the distance between them, $F = k/s^2$. If one electron is held fixed at the origin, find the work required to move a second electron along the x-axis from the point (10, 0) to the point (5, 0).

43 If one electron is held fixed at the point (0, 0) and another at the point (100, 0), find the work required to move a third electron along the x-axis from (50, 0) to (80, 0).

6.7 IMPROPER INTEGRALS

What is the area of the region under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 6.7.1(a))? The function $1/\sqrt{x}$ is not continuous at $x = 0$, and in fact $1/\sqrt{\varepsilon}$ is infinite for infinitesimal $\varepsilon > 0$. Thus our notion of a definite integral does not apply. Nevertheless we shall be able to assign an area to the region using improper integrals. We see from the figure that the region extends infinitely far up in the vertical direction. However, it becomes so thin that the area of the region turns out to be finite.

The region of Figure 6.7.1(b) under the curve $y = x^{-3}$ from $x = 1$ to $x = \infty$

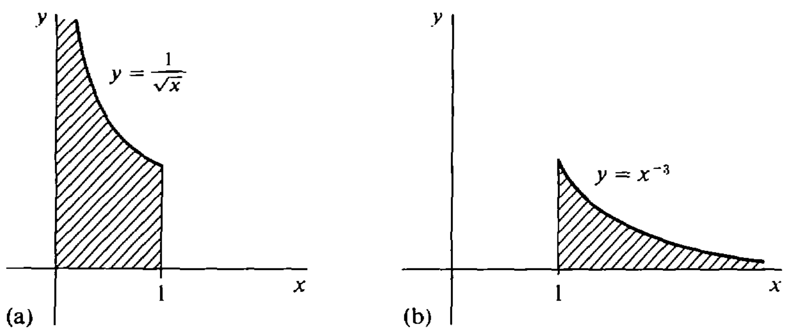


Figure 6.7.1

extends infinitely far in the horizontal direction. We shall see that this region, too, has a finite area which is given by an improper integral.

Improper integrals are defined as follows.

DEFINITION

Suppose f is continuous on the half-open interval $(a, b]$. The ***improper integral*** *of f from a to b is defined by the limit*

$$\int_a^b f(x)\,dx = \lim_{u \to a^-} \int_u^b f(x)\,dx.$$

If the limit exists the improper integral is said to converge. Otherwise the improper integral is said to diverge.

The improper integral can also be described in terms of definite integrals with hyperreal endpoints. We first recall that the definite integral

$$D(u, v) = \int_u^v f(x)\,dx$$

is a real function of two variables u and v. If u and v vary over the hyperreal numbers instead of the real numbers, the definite integral $\int_u^v f(x)\,dx$ stands for the natural extension of D evaluated at (u, v),

$$D^*(u, v) = \int_u^v f(x)\,dx.$$

Here is the description of the improper integral using definite integrals with hyperreal endpoints.

Let f be continuous on $(a, b]$.

(1) *$\int_a^b f(x)\,dx = S$ if and only if $\int_{a+\varepsilon}^b f(x)\,dx \approx S$ for all positive infinitesimal ε.*

(2) *$\int_a^b f(x)\,dx = \infty$ (or $-\infty$) if and only if $\int_{a+\varepsilon}^b f(x)\,dx$ is positive infinite (or negative infinite) for all positive infinitesimal ε.*

EXAMPLE 1 Find $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$. For $u > 0$,

$$\int_u^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_u^1 = 2 - 2\sqrt{u}.$$

Then $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{u \to 0^+} \int_u^1 \frac{1}{\sqrt{x}}\,dx = \lim_{u \to 0^+} (2 - 2\sqrt{u}) = 2.$$

Therefore the region under the curve $y = 1/\sqrt{x}$ from 0 to 1 shown in Figure 6.7.1(a) has area 2, and the improper integral converges.

EXAMPLE 2 Find $\int_0^1 x^{-2}\,dx$. For $u > 0$,

$$\int_u^1 x^{-2}\,dx = -x^{-1}\Big]_u^1 = -1 + \frac{1}{u}.$$

This time $$\lim_{u\to 0^+}\int_u^1 x^{-2}\,dx = \lim_{u\to 0^+}\left(-1+\frac{1}{u}\right) = \infty.$$

The improper integral diverges. Since the limit goes to infinity we may write

$$\int_0^1 x^{-2}\,dx = \infty.$$

The region under the curve in Figure 6.7.2 is said to have *infinite area.*

Warning: We remind the reader once again that the symbols ∞ and $-\infty$ are not real or even hyperreal numbers. We use them only to indicate the behavior of a limit, or to indicate an interval without an upper or lower endpoint.

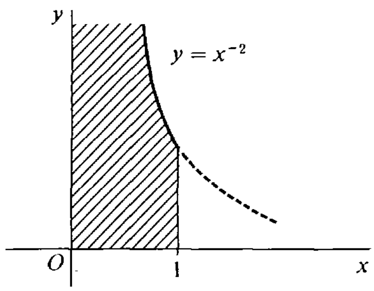


Figure 6.7.2

EXAMPLE 3 Find the length of the curve $y = x^{2/3}$, $0 \le x \le 8$. From Figure 6.7.3 the curve must have finite length. However, the derivative

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$$

is undefined at $x = 0$. Thus the length formula gives an improper integral,

$$s = \int_0^8 \sqrt{1 + (dy/dx)^2}\,dx = \int_0^8 \sqrt{1 + \tfrac{4}{9}x^{-2/3}}\,dx = \int_0^8 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\,dx$$

$$= \int_0^8 \frac{1}{3x^{1/3}}\sqrt{9x^{2/3}+4}\,dx = \lim_{a\to 0^+}\int_a^8 \frac{1}{3x^{1/3}}\sqrt{9x^{2/3}+4}\,dx.$$

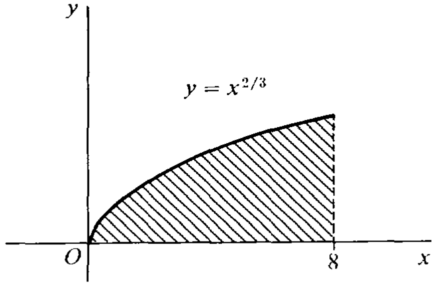


Figure 6.7.3

Let $u = 9x^{2/3} + 4$, $du = 6x^{-1/3}\,dx$. The indefinite integral is

$$\int \frac{1}{3x^{1/3}}\sqrt{9x^{2/3} + 4}\,dx = \int \frac{1}{18}\sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{27}(9x^{2/3} + 4)^{3/2} + C.$$

Therefore

$$s = \lim_{a \to 0^+} \frac{1}{27}(9x^{2/3} + 4)^{3/2}\Big]_a^8$$

$$= \frac{1}{27}((9\cdot 4 + 4)^{3/2} - (9\cdot 0 + 4)^{3/2}) = \frac{8}{27}(10\sqrt{10} - 1).$$

Notice that we use the same symbol for both the definite and the improper integral. The theorem below justifies this practice.

THEOREM 1

If f is continuous on the closed interval $[a, b]$ then the improper integral of f from a to b converges and equals the definite integral of f from a to b.

PROOF We have shown in Section 4.2 on the Fundamental Theorem that the function

$$F(u) = \int_u^b f(x)\,dx$$

is continuous on $[a, b]$. Therefore

$$\int_a^b f(x)\,dx = \lim_{u \to a^+} \int_u^b f(x)\,dx,$$

where $\int_a^b f(x)\,dx$ denotes the definite integral.

We now define a second kind of improper integral where the interval is infinite.

DEFINITION

Let f be continuous on the half-open interval $[a, \infty)$. The ***improper integral*** *of f from a to ∞ is defined by the limit*

$$\int_a^\infty f(x)\,dx = \lim_{u \to \infty} \int_a^u f(x)\,dx.$$

The improper integral is said to converge if the limit exists and to diverge otherwise.

Here is a description of this kind of improper integral using definite integrals with hyperreal endpoints.

Let f be continuous on $[a, \infty)$.

(1) *$\int_a^\infty f(x)\,dx = S$ if and only if $\int_a^H f(x)\,dx \approx S$ for all positive infinite H.*

(2) $\int_a^\infty f(x)\,dx = \infty$ *(or* $-\infty$*) if and only if* $\int_a^H f(x)\,dx$ *is positive infinite (or negative infinite) for all positive infinite* H.

EXAMPLE 4 Find the area under the curve $y = x^{-3}$ from 1 to ∞. The area is given by the improper integral

$$\int_1^\infty x^{-3}\,dx.$$

For $u > 0$, $\int_1^u x^{-3}\,dx = -\tfrac{1}{2}x^{-2}\Big]_1^u = -\tfrac{1}{2}u^{-2} + \tfrac{1}{2}.$

Thus $$\int_1^\infty x^{-3}\,dx = \lim_{u\to\infty} \int_1^u x^{-3}\,dx = \lim_{u\to\infty} (-\tfrac{1}{2}u^{-2} + \tfrac{1}{2}) = \tfrac{1}{2}.$$

So the improper integral converges and the region has area $\frac{1}{2}$. The region is shown in Figure 6.7.1(b) and extends infinitely far to the right.

EXAMPLE 5 Find the area under the curve $y = x^{-2/3}$, $1 \le x < \infty$.

$$A = \int_1^\infty x^{-2/3}\,dx = \lim_{u\to\infty} \int_1^u x^{-2/3}\,dx$$

$$= \lim_{u\to\infty} 3x^{1/3}\Big]_1^u = \lim_{u\to\infty} 3(u^{1/3} - 1) = \infty.$$

The region is shown in Figure 6.7.4 and has infinite area.

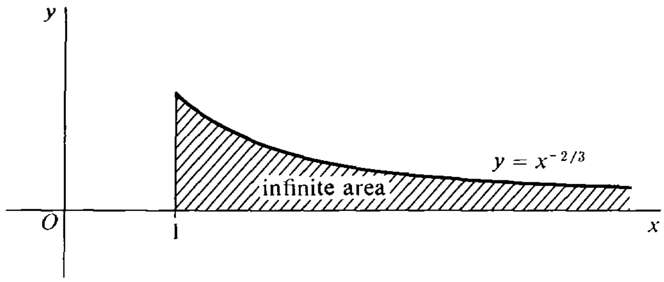


Figure 6.7.4

EXAMPLE 6 The region in Example 5 is rotated about the x-axis. Find the volume of the solid of revolution.

We use the Disc Method because the rotation is about the axis of the independent variable. The volume formula gives us an improper integral.

$$V = \int_1^\infty \pi(x^{-2/3})^2\,dx = \int_1^\infty \pi x^{-4/3}\,dx$$

$$= \lim_{u\to\infty} \int_1^u \pi x^{-4/3}\,dx = \lim_{u\to\infty} -3\pi x^{-1/3}\Big]_1^u$$

$$= \lim_{u\to\infty} 3\pi(-u^{-1/3} + 1) = 3\pi.$$

So the solid shown in Figure 6.7.5 has finite volume $V = 3\pi$.

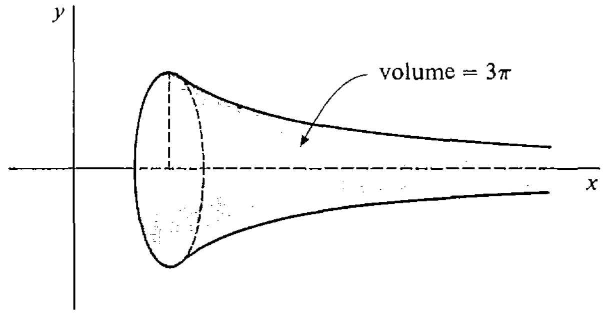


Figure 6.7.5

The last two examples give an unexpected result. A region with *infinite* area is rotated about the x-axis and generates a solid with *finite* volume! In terms of hyperreal numbers, the area of the region under the curve $y = x^{-2/3}$ from 1 to an infinite hyperreal number H is equal to $3(H^{1/3} - 1)$, which is positive infinite. But the volume of the solid of revolution from 1 to H is equal to

$$3\pi(1 - H^{-1/3}),$$

which is finite and has standard part 3π.

We can give a simpler example of this phenomenon. Let H be a positive infinite hyperinteger, and form a cylinder of radius $1/H$ and length H^2 (Figure 6.7.6). Then the cylinder is formed by rotating a rectangle of length H^2, width $1/H$, and infinite area $H^2/H = H$. But the volume of the cylinder is equal to π,

$$V = \pi r^2 h = \pi(1/H)^2(H^2) = \pi.$$

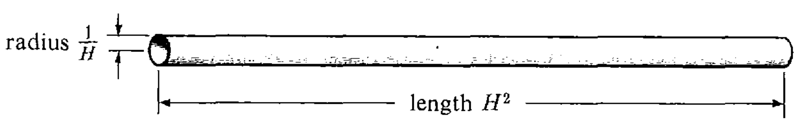


Figure 6.7.6 Area = H, volume = π

Imagine a cylinder made out of modelling clay, with initial length and radius one. The volume is π. The clay is carefully stretched so that the cylinder gets longer and thinner. The volume stays the same, but the area of the cross section keeps getting bigger. When the length becomes infinite, the cylinder of clay still has finite volume $V = \pi$, but the area of the cross section has become infinite.

There are other types of improper integrals. If f is continuous on the half-open interval $[a, b)$ then we define

$$\int_a^b f(x)\,dx = \lim_{u \to b^-} \int_a^u f(x)\,dx.$$

If f is continuous on $(-\infty, b]$ we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{u \to -\infty} \int_u^b f(x)\,dx.$$

We have introduced four types of improper integrals corresponding to the four types of half-open intervals

$$[a, b), \qquad [a, \infty), \qquad (a, b], \qquad (-\infty, b].$$

By piecing together improper integrals of these four types we can assign an improper integral to most functions which arise in calculus.

DEFINITION

A function f is said to be ***piecewise continuous*** *on an interval I if f is defined and continuous at all but perhaps finitely many points of I. In particular, every continuous function is piecewise continuous.*

We can introduce the improper integral $\int_a^b f(x)\,dx$ whenever f is piecewise continuous on I and a, b are either the endpoints of I or the appropriate infinity symbol. A few examples will show how this can be done.

Let f be continuous at every point of the closed interval $[a, b]$ except at one point c where $a < c < b$. We define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

EXAMPLE 7 Find the improper integral $\int_{-8}^{1} x^{-1/3}\,dx$. $x^{-1/3}$ is discontinuous at $x = 0$. The indefinite integral is

$$\int x^{-1/3}\,dx = \tfrac{3}{2}x^{2/3} + C.$$

Then
$$\int_{-8}^{0} x^{-1/3}\,dx = \lim_{u \to 0^-} \int_{-8}^{u} x^{-1/3}\,dx = \lim_{u \to 0^-} \tfrac{3}{2}x^{2/3}\Big]_{-8}^{u}$$
$$= \lim_{u \to 0^-} \left(\tfrac{3}{2}u^{2/3} - \tfrac{3}{2}(-8)^{2/3}\right) = -\tfrac{3}{2}\cdot 4 = -6.$$

Similarly,
$$\int_0^1 x^{-1/3}\,dx = \tfrac{3}{2}.$$

So
$$\int_{-8}^{1} x^{-1/3}\,dx = -6 + \tfrac{3}{2} = -\tfrac{9}{2}$$

and the improper integral converges. Thus, the region shown in Figure 6.7.7 has finite area.

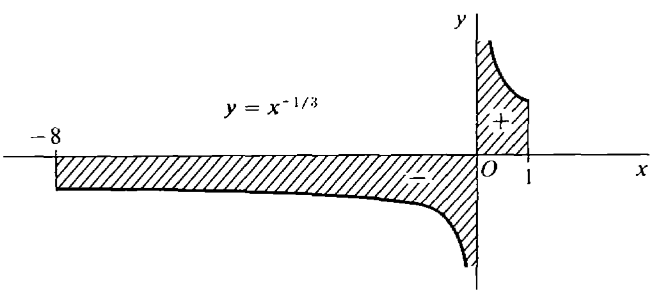


Figure 6.7.7

If f is continuous on the *open* interval (a, b), the improper integral is defined as the sum

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where c is any point in the interval (a, b). The endpoints a and b may be finite or infinite. It does not matter which point c is chosen, because if e is any other point in (a, b), then

$$\begin{aligned}\int_a^c f(x)\,dx + \int_c^b f(x)\,dx &= \int_a^c f(x)\,dx + \left(\int_c^e f(x)\,dx + \int_e^b f(x)\,dx\right)\\ &= \left(\int_a^c f(x)\,dx + \int_c^e f(x)\,dx\right) + \int_e^b f(x)\,dx\\ &= \int_a^e f(x)\,dx + \int_e^b f(x)\,dx.\end{aligned}$$

EXAMPLE 8 Find $\displaystyle\int_0^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx.$

The function $2/\sqrt{x} + 1/\sqrt{2-x}$ is continuous on the open interval $(0, 2)$ but discontinuous at both endpoints (Figure 6.7.8). Thus

$$\int_0^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx = \int_0^1 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx + \int_1^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx.$$

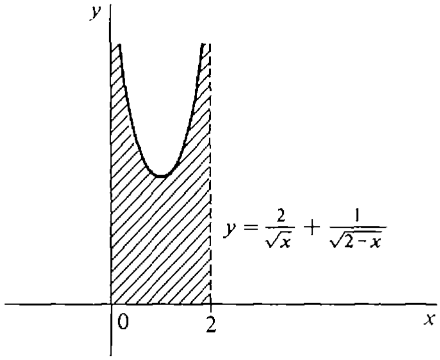


Figure 6.7.8

First we find the indefinite integral.

$$\int \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx = 4\sqrt{x} - 2\sqrt{2-x} + C.$$

Then $\displaystyle\int_0^1 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx = \lim_{u\to 0^+}\int_u^1 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx$

$$= \lim_{u\to 0^-}(4\sqrt{x} - 2\sqrt{2-x})\Big]_u^1$$

$$= (4-2) - (0 - 2\sqrt{2}) = 2 + 2\sqrt{2}.$$

Also $$\int_1^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx = \lim_{v\to 2^-}\int_1^v \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx$$
$$= \lim_{v\to 2^-} (4\sqrt{x} - 2\sqrt{2-x})\Big]_1^v$$
$$= (4\sqrt{2} - 0) - (4 - 2) = 4\sqrt{2} - 2.$$

Therefore $$\int_0^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}\,dx = (2 + 2\sqrt{2}) + (4\sqrt{2} - 2) = 6\sqrt{2}.$$

EXAMPLE 9 Find $\displaystyle\int_0^1 \frac{1}{x^2} + \frac{1}{(x-1)^2}\,dx.$

The function $1/x^2 + 1/(x-1)^2$ is continuous on the open interval (0, 1) but discontinuous at both endpoints. The indefinite integral is

$$\int \frac{1}{x^2} + \frac{1}{(x-1)^2}\,dx = -\frac{1}{x} - \frac{1}{x-1} + C.$$

We have $$\int_0^{1/2} \frac{1}{x^2} + \frac{1}{(x-1)^2}\,dx = \lim_{u\to 0^+}\left(-\frac{1}{x} - \frac{1}{x-1}\right)\Big]_u^{1/2}$$
$$= \lim_{u\to 0^+}\left(\frac{1}{u} + \frac{1}{u-1}\right) = \infty.$$

Similarly we find that

$$\int_{1/2}^1 \frac{1}{x^2} + \frac{1}{(x-1)^2}\,dx = \infty.$$

In this situation we may write

$$\int_0^1 \frac{1}{x^2} + \frac{1}{(x-1)^2}\,dx = \infty,$$

and we say that the region under the curve in Figure 6.7.9 has infinite area.

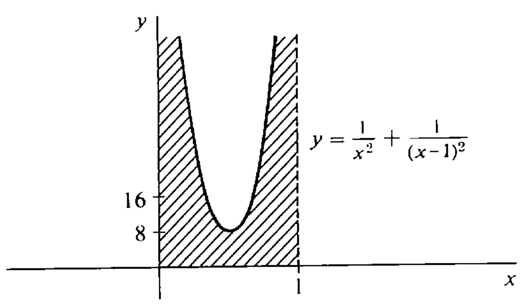


Figure 6.7.9

Remark In Example 9 we are faced with a sum of two infinite limits. Using the rules for adding infinite hyperreal numbers as a guide we can give rules for sums of infinite limits.

If H and K are positive infinite hyperreal numbers and c is finite, then

$$H + K \text{ is positive infinite,}$$

$$H + c \text{ is positive infinite,}$$

$$-H - K \text{ is negative infinite,}$$

$$-H + c \text{ is negative infinite,}$$

$H - K$ can be either finite, positive infinite, or negative infinite.

By analogy, we use the following rules for sums of two infinite limits or of a finite and an infinite limit. These rules tell us when such a sum can be considered to be positive or negative infinite. We use the infinity symbols as a convenient shorthand, keeping in mind that they are not even hyperreal numbers.

$$\infty + \infty = \infty,$$

$$\infty + c = \infty,$$

$$-\infty - \infty = -\infty,$$

$$-\infty + c = -\infty,$$

$$\infty - \infty \text{ is undefined.}$$

EXAMPLE 10 Find $\int_{-\infty}^{\infty} x\,dx$. We see that

$$\int_{-\infty}^{0} x\,dx = \lim_{u \to -\infty} \int_{u}^{0} x\,dx = \lim_{u \to -\infty} \tfrac{1}{2}x^2\Big]_u^0 = -\infty,$$

and

$$\int_{0}^{\infty} x\,dx = \infty.$$

Thus $\int_{-\infty}^{\infty} x\,dx$ diverges and has the form $\infty - \infty$. We do not assign it any value or either of the symbols ∞ or $-\infty$. The region under the curve $f(x) = x$ is shown in Figure 6.7.10.

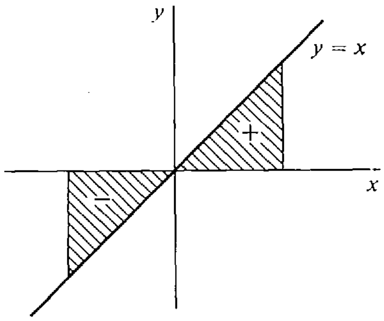


Figure 6.7.10

It is tempting to argue that the positive area to the right of the origin and the negative area to the left exactly cancel each other out so that the improper integral is zero. But this leads to a paradox.

Wrong: $\int_{-\infty}^{\infty} x\,dx = 0$. Let $v = x + 2$, $dv = dx$. Then

$$\int_{-\infty}^{\infty} (x + 2)\,dx = \int_{-\infty}^{\infty} v\,dv = 0.$$

Subtracting $\int_{-\infty}^{\infty} (x+2) - x\,dx = 0 - 0 = 0, \qquad \int_{-\infty}^{\infty} 2\,dx = 0.$

But $\int_{-\infty}^{\infty} 2\,dx = \infty.$

So we do not give the integral $\int_{-\infty}^{\infty} x\,dx$ the value 0, and instead leave it undefined.

PROBLEMS FOR SECTION 6.7

In Problems 1–36, test the improper integral for convergence and evaluate when possible.

1 $\int_2^\infty x^{-2}\,dx$

2 $\int_0^1 x^{-0.9}\,dx$

3 $\int_1^\infty x^{-1/2}\,dx$

4 $\int_{-\infty}^0 (2x-1)^{-3}\,dx$

5 $\int_0^{1/2} (2x-1)^{-3}\,dx$

6 $\int_{-1}^0 x^{-1/3}\,dx$

7 $\int_0^\infty x^2 + 2x - 1\,dx$

8 $\int_3^\infty x^{-2} - x^{-3}\,dx$

9 $\int_0^\infty x(1+x^2)^{-2}\,dx$

10 $\int_1^\infty x^{-1/2} + x^{-2}\,dx$

11 $\int_0^1 x^{-1/2} + x^{-2}\,dx$

12 $\int_0^1 (1-x)^{-1/2}\,dx$

13 $\int_0^1 (x-1)^{-2/3}\,dx$

14 $\int_{-1}^1 x^{-2}\,dx$

15 $\int_{-1}^1 x^{-2/3}\,dx$

16 $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

17 $\int_0^1 2x(x^2-1)^{-1/3}\,dx$

18 $\int_{-1}^1 2x^{-3}\,dx$

19 $\int_0^1 (2x-1)^{-2/3}\,dx$

20 $\int_0^1 (3x-1)^{-5}\,dx$

21 $\int_{-\infty}^\infty x^2\,dx$

22 $\int_{-\infty}^\infty (2x-1)^3\,dx$

23 $\int_0^\infty \frac{1}{\sqrt{x}}\,dx$

24 $\int_{-\infty}^\infty x^{-1/3}\,dx$

25 $\int_{-\infty}^\infty x^3\,dx$

26 $\int_0^\infty x^{-3/2}\,dx$

27 $\int_0^\infty \frac{3x}{(x+1)^4}\,dx$

28 $\int_{-\infty}^\infty |x|(x^2+1)^{-3}\,dx$

29 $\int_{-\infty}^\infty \frac{2x}{\sqrt{x^2+1}}\,dx$

30 $\int_1^3 (x-1)^{-2} + (x-3)^{-2}\,dx$

31 $\int_1^3 (x-1)^{-1/2} + (3-x)^{-1/2}\,dx$

32 $\int_{-5}^3 \frac{x}{|x|}\,dx$

33 $\int_0^{\pi/2} \frac{\sin\theta}{\cos^2\theta}\,d\theta$

34 $\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{\cos\theta}}\,d\theta$

35 $\int_0^{10} f(x)\,dx \quad \text{where } f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 6 & \text{for } 1 \le x < 5 \\ 3 & \text{for } 5 \le x \le 10 \end{cases}$

36 $\int_0^{\infty} f(x)\,dx \quad \text{where } f(x) = \begin{cases} 1/\sqrt{x} & \text{if } 0 < x < 1 \\ x^{-2} & \text{if } 1 \le x \end{cases}$

37 Show that if r is a rational number, the improper integral $\int_0^{100} x^{-r}\,dx$ converges when $r < 1$ and diverges when $r > 1$.

38 Show that if r is rational, the improper integral $\int_{1/2}^{\infty} x^{-r}\,dr$ converges when $r > 1$ and diverges when $r < 1$.

39 Find the area of the region under the curve $y = 4x^{-2}$ from $x = 1$ to $x = \infty$.

40 Find the area of the region under the curve $y = 1/\sqrt{2x - 1}$ from $x = \frac{1}{2}$ to $x = 1$.

41 Find the area of the region between the curves $y = x^{-1/4}$ and $y = x^{-1/2}$ from $x = 0$ to $x = 1$.

42 Find the area of the region between the curves $y = -x^{-3}$ and $y = x^{-2}$, $1 \le x < \infty$.

43 Find the volume of the solid generated by rotating the curve $y = 1/x$, $1 \le x < \infty$, about (a) the x-axis, (b) the y-axis.

44 Find the volume of the solid generated by rotating the curve $y = x^{-1/3}$, $0 < x \le 1$, about (a) the x-axis, (b) the y-axis.

45 Find the volume of the solid generated by rotating the curve $y = x^{-3/2}$, $0 < x \le 4$, about (a) the x-axis, (b) the y-axis.

46 Find the volume generated by rotating the curve $y = 4x^{-3}$, $-\infty < x \le -2$, about (a) the x-axis, (b) the y-axis.

47 Find the length of the curve $y = \sqrt{x} - \frac{1}{3}x\sqrt{x}$ from $x = 0$ to $x = 1$.

48 Find the length of the curve $y = \frac{3}{4}x^{1/3} - \frac{3}{5}x^{5/3}$ from $x = 0$ to $x = 1$.

49 Find the surface area generated when the curve $y = \sqrt{x} - \frac{1}{3}x\sqrt{x}$, $0 \le x \le 1$, is rotated about (a) the x-axis, (b) the y-axis.

50 Do the same for the curve $y = \frac{3}{4}x^{1/3} - \frac{3}{5}x^{5/3}$, $0 \le x \le 1$.

51 (a) Find the surface area generated by rotating the curve $y = \sqrt{x}$, $0 \le x \le 1$, about the x-axis.
(b) Set up an integral for the area generated about the y-axis.

52 Find the surface area generated by rotating the curve $y = x^{2/3}$, $0 \le x \le 8$, about the x-axis.

53 Find the surface area generated by rotating the curve $y = \sqrt{r^2 - x^2}$, $0 \le x \le a$, about (a) the x-axis, (b) the y-axis $(0 < a \le r)$.

54 The force of gravity between particles of mass m_1 and m_2 is $F = gm_1m_2/s^2$ where s is the distance between them. If m_1 is held fixed at the origin, find the work done in moving m_2 from the point $(1, 0)$ all the way out the x-axis.

☐ 55 Show that the Rectangle and Addition Properties hold for improper integrals.

EXTRA PROBLEMS FOR CHAPTER 6

1 The skin is peeled off a spherical apple in four pieces in such a way that each horizontal cross section is a square whose corners are on the original surface of the apple. If the original apple had radius r, find the volume of the peeled apple.

2 Find the volume of a tetrahedron of height h and base a right triangle with legs of length a and b.

3 Find the volume of the wedge formed by cutting a right circular cylinder of radius r with two planes, meeting on a line crossing the axis, one plane perpendicular to the axis and the other at a 45° angle.

4 Find the volume of a solid whose base is the region between the x-axis and the curve $y = 1 - x^2$, and which intersects each plane perpendicular to the x-axis in a square.

In Problems 5–8, the region bounded by the given curves is rotated about (a) the x-axis, (b) the y-axis. Find the volumes of the two solids of revolution.

5 $y = 0, \quad y = \sqrt{4 - x^2}, \quad 0 \le x \le 1$

6 $y = 0, \quad y = x^{3/2}, \quad 0 \le x \le 1$

7 $y = x, \quad y = 4 - x, \quad 0 \le x \le 2$

8 $y = x^p, \quad y = x^q, \quad 0 \le x \le 1, \quad \text{where } 0 < q < p$

9 The region under the curve $y = \sqrt{1 - x^p}$, $0 \le x \le 1$, where $0 < p$, is rotated about the x-axis. Find the volume of the solid of revolution.

10 The region under the curve $y = (x^2 + 4)^{1/3}, 0 \le x \le 2$, is rotated about the y-axis. Find the volume of the solid of revolution.

11 Find the length of the curve $y = (2x + 1)^{3/2}, 0 \le x \le 2$.

12 Find the length of the curve $y = 3x - 2, 0 \le x \le 4$.

13 Find the length of the curve $x = 3t + 1, y = 2 - 4t, 0 \le t \le 1$.

14 Find the length of the curve $x = f(t), y = f(t) + c, a \le t \le b$.

15 Find the length of the line $x = At + B, y = Ct + D, a \le t \le b$.

16 Find the area of the surface generated by rotating the curve $y = 3x^2 - 2, 0 \le x \le 1$, about the y-axis.

17 Find the area of the surface generated by rotating the curve $x = At^2 + Bt, y = 2At + B$, $0 \le t \le 1$, about the x-axis. $A > 0, B > 0$.

18 Find the average value of $f(x) = x/\sqrt{x^2 + 1}, 0 \le x \le 4$.

19 Find the average value of $f(x) = x^p, 1 \le x \le b, p \ne -1$.

20 Find the average distance from the origin of a point on the parabola $y = x^2, 0 \le x \le 4$, with respect to x.

21 Given that $f(x) = x^p, 0 \le x \le 1$, p a positive constant, find a point c between 0 and 1 such that $f(c)$ equals the average value of $f(x)$.

22 Find the center of mass of a wire on the x-axis, $0 \le x \le 2$, whose density at a point x is equal to the square of the distance from $(x, 0)$ to $(0, 1)$.

23 Find the center of mass of a length of wire with constant density bent into three line segments covering the top, left, and right edges of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$.

24 Find the center of mass of a plane object bounded by the lines $y = 0, y = x, x = 1$, with density $\rho(x) = 1/x$.

25 Find the center of mass of a plane object bounded by the curves $x = y^2, x = 1$, with density $\rho(y) = y^2$.

26 Find the centroid of the triangle bounded by the x- and y-axes and the line $ax + by = c$, where a, b, and c are positive constants.

27 A spring exerts a force of $10x$ lbs when stretched a distance x beyond its natural length of 2 ft. Find the work required to stretch the spring from a length of 3 ft to 4 ft.

In Problems 28–36, test the improper integral for convergence and evaluate if it converges.

28 $\int_{-\infty}^{-2} x^{-3}\, dx$

29 $\int_0^{\infty} (x + 2)^{-1/4}\, dx$

30 $\int_{-1}^{0} x^{-4}\,dx$

31 $\int_{-1}^{0} x^{-1/5}\,dx$

32 $\int_{-\infty}^{\infty} x^{1/5}\,dx$

33 $\int_{0}^{1} \frac{1}{x^2} + \frac{1}{(x-1)^2}\,dx$

34 $\int_{0}^{1} \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\,dx$

35 $\int_{-4}^{4} \frac{1}{\sqrt{|x|}}\,dx$

36 $\int_{0}^{\infty} \sin x\,dx$

37 A wire has the shape of a curve $y = f(x)$, $a \le x \le b$, and has density $\rho(x)$ at value x. Justify the formulas below for the mass and moments of the wire.

$$m = \int_a^b \rho(x)\sqrt{1 + (f'(x))^2}\,dx,$$

$$M_x = \int_a^b f(x)\rho(x)\sqrt{1 + (f'(x))^2}\,dx,$$

$$M_y = \int_a^b x\rho(x)\sqrt{1 + (f'(x))^2}\,dx.$$

38 Find the mass, moments, and center of mass of a wire bent in the shape of a parabola $y = x^2$, $-1 \le x \le 1$, with density $\rho(x) = \sqrt{1 + 4x^2}$.

39 Find the mass, moments, and center of mass of a wire of constant density ρ bent in the shape of the semicircle $y = \sqrt{1 - x^2}$, $-1 \le x \le 1$.

☐ 40 An object fills the solid generated by rotating the region under the curve $y = f(x)$, $a \le x \le b$, about the x-axis. Its density per unit volume is $\rho(x)$. Justify the following formula for the mass of the object.

$$m = \int_a^b \rho(x)\pi(f(x))^2\,dx.$$

☐ 41 A container filled with water has the shape of a solid of revolution formed by rotating the curve $x = g(y)$, $a \le y \le b$, about the (vertical) y-axis. Water has constant density ρ per unit volume. Justify the formula below for the amount of work needed to pump all the water to the top of the container.

$$W = \int_a^b \rho\pi(g(y))^2(b - y)\,dy.$$

42 Find the work needed to pump all the water to the top of a water-filled container in the shape of a cylinder with height h and circular base of radius r.

43 Do Problem 46 if the container is in the shape of a hemispherical bowl of radius r.

44 Do Problem 46 if the container is in the shape of a cone with its vertex at the bottom, height h, and circular top of radius r.

☐ 45 The *pressure*, or force per unit area, exerted by water on the walls of a container is equal to $p = \rho(b - y)$ where ρ is the density of water and $b - y$ the water depth. Find the total force on a dam in the shape of a vertical rectangle of height b and width w, assuming the water comes to the top of the dam.

☐ 46 A water-filled container has the shape of a solid formed by rotating the curve $x = g(y)$, $a \le y \le b$ about the (vertical) y-axis. Justify the formula below for the total force on the walls of the container.

$$F = \int_a^b 2\pi\rho(b - y)x\sqrt{(dx/dy)^2 + 1}\,dy$$